GOVERNANCE AND THE EFFICIENCY OF ECONOMIC SYSTEMS

GESY

Discussion Paper No. 351

How to allocate Research (and other) Subsidies

Ludwig Ensthaler*

Thomas Giebe**

* Department of Economics, Humboldt University at Berlin, School of Business and Economics, Spandauer Straße 1, 10178 Berlin, Germany, and Graduate Center of Economic and Social Research DIW Berlin

E-mail: lensthaler@diw.de

** Department of Economics, Humboldt University at Berlin, School of Business and Economics, Spandauer Straße 1, 10178 Berlin, Germany

E-mail: thomas.giebe@wiwi.hu-berlin.de

March 2011

Financial support from the Deutsche Forschungsgemeinschaft through SFB/TR 15 is gratefully acknowledged.

Sonderforschungsbereich/Transregio 15 · www.sfbtr15.de

Universität Mannheim · Freie Universität Berlin · Humboldt-Universität zu Berlin · Ludwig-Maximilians-Universität München

Rheinische Friedrich-Wilhelms-Universität Bonn · Zentrum für Europäische Wirtschaftsforschung Mannheim

Speaker: Prof. Dr. Urs Schweizer. · Department of Economics · University of Bonn · D-53113 Bonn,

Phone: +49(0228)739220 · Fax: +49(0228)739221

How to allocate Research (and other) Subsidies*

Ludwig Ensthaler†
Humboldt University at Berlin
DIW Berlin

Thomas Giebe‡
Humboldt University at Berlin

March 4, 2011

Abstract

A budget-constrained buyer wants to purchase items from a short-listed set. Items are differentiated by observable quality and sellers have private reserve prices for their items. The buyer's problem is to select a subset of maximal quality. Money does not enter the buyer's objective function, but only his constraints. Sellers quote prices strategically, inducing a knapsack game. We derive the Bayesian optimal mechanism for the buyer's problem. We find that simultaneous take-it-or-leave-it offers are optimal. Hence, somewhat surprisingly, ex-post competition is not required to implement optimality. Finally, we discuss the problem in a detail free setting.

JEL *D21, D44, D45, D82.*

Keywords: *Mechanism Design, Subsidies, Budget, Procurement, Knapsack Problem*

*Many thanks to Tilman Börgers, Jeff Ely, Kurt Helmes, Ulrich Kamecke, Phil Reny, Rakesh Vohra, Georg Weizsäcker, and Elmar Wolfstetter for helpful comments. Financial support from the Deutsche Forschungsgemeinschaft through SFB/TR 15 is gratefully acknowledged. Parts of this paper were written while the first author was a visiting predoc at the Center for Mathematical Studies in Economics and Management Science at NU Kellogg MEDS. Hospitality by Kellogg School of Management and Northwestern University is gratefully acknowledged.

†Humboldt University at Berlin, School of Business and Economics, Dept. of Economics, Spandauer Straße 1, 10178 Berlin, Germany, and Graduate Center of Economic and Social Research DIW Berlin, Tel: +49-30-2093-1594, Fax: +49-30-2093-5619, `lensthaler@diw.de`

‡(Corresponding author) Humboldt University at Berlin, School of Business and Economics, Dept. of Economics, Spandauer Straße 1, 10178 Berlin, Germany, Tel: +49-30-2093-5773, Fax: +49-30-2093-5619, `thomas.giebe@wiwi.hu-berlin.de`.

Consider a buyer who has a fixed budget to spend on items from a shortlisted set. The items differ in quality. The quality of a subset of items is the sum of the individual qualities of its elements. A subset of higher quality is preferred to one of lower quality. Subsets of the same quality are considered as perfect substitutes. Each seller has private information about his reserve price for his item. The buyer's problem is to select a subset of items of maximal quality subject to his budget constraint.

Under complete information, the buyer faces a binary knapsack problem with qualities corresponding to values and reserve prices corresponding to weights in the standard notation. In the realm of incomplete information, any buying mechanism induces a knapsack game where sellers choose the weight of their item (i.e. the price they quote) strategically.

For an important application of this problem, consider government funds to subsidize R&D activities by private businesses. Typically, an agency has a fixed budget to spend to support research projects. Researchers apply for grants by submitting both a detailed plan of the research to be conducted and the associated costs. The quality of the proposals is then evaluated by a panel of independent experts. Based on these evaluations and on the stated costs the agency makes a funding offer. The agency's objective is to maximize the total quality of the supported projects.

What makes this problem non-standard is the procurer's objective, which requires special attention. In most other procurement problems money is part of the procurers objective function in the sense that the procurers welfare depends directly on the prices at which procurement happens. Not so here. Program managers do not value money in the sense that they assign a marginal value to it. Rather, they will support an additional project as long as they have enough money to do so. In other words, for them, there is no tradeoff between funding a shortlisted project and keeping the money. In fact, many public funding agencies are under pressure to spend all the money they have in one year in order to avoid a budget cut in the next. Hence, when we model this setting, money does only enter the procurer's constraints, not the objective function.

The surprising fact from an economic point of view is that the agencies usually do not enforce financial competition among the proposers. Instead, once all projects have been evaluated by the panel of experts, the agencies make multiple, simultaneous take-it-or-leave-it offers to the proposers. However, one could think that it would be better to engage proposers in a price competition. For example, as often the case in procurement, the agency could award to the projects with the best quality-per-money score. In this paper we show that the procedure we observe in the real world, that is, making simultaneous take-it-or-leave-it offers might indeed be optimal and that price competition is not needed to implement the optimal expected allocation.

While the analysis of subsidy schemes for R&D projects is the main motivation for this paper, we would like to mention that one can easily think of

many other applications, for example scholarships and bursaries for students and emissions abatement (see below).
We proceed as follows. Section 1 introduces the formal model. In section 2, we derive an optimal Bayesian mechanism for the problem. In section 3, we analyse the problem in a detail free setting, i.e. when sellers' distributions are not assumed to be public knowledge. We discuss an intuitive auction mechanism which favourable properties in that setting. Section 4 concludes.

Literature A similar allocation problem in the context of R&D subsidies was first studied by Giebe et al. (2006). They point out flaws in the widely applied rules for awarding R&D subsidies. Among several recommendations, they experimentally study the performance of open auctions as a means of inducing competition for funding. These open auctions are related to the auction that we analyze in a belief free setting in section 3.
Despite its practical relevance, the problem has received surprisingly little attention in the mechanism design literature. To the best of our knowledge, there is no published paper on this issue. There are, however, two related unpublished manuscripts: In his Nancy L. Schwartz lecture, Eric Maskin (2010) analyses the UK emissions reduction auction. In this auction, the UK government spent a predetermined fixed fund to pay firms to cut CO_2 emissions. Since firms abatement costs are private information, this is a mechanism design problem. Maskin proceeds to derive the optimal ex-post mechanism (that is, a mechanism which satisfies ex-post IC, ex-post IR and ex-post budget balance) for special classes of distributions.
In an unpublished response, and independent to this work, Chung and Ely (2002) derive the optimal Bayesian (interim) mechanism. The results, where applicable, coincide with the ones presented in this paper. In particular, Chung and Ely also find that the optimal mechanism does not require ex-post competition. However, in this paper we derive the optimal mechanism more explicitly and in a constructive way.
Papadimitriou and Singer (2010), define a whole new class of mechanism design problems that includes our problem as a special case. They call mechanisms which have to satisfy a budget constraint as the one present in this problem *budget feasible mechanism*. The authors, however, focus on worst-case analysis rather than Bayesian analysis of the problem.
On the more technical side, our problem is a game-theoretic variant of the knapsack problem (see e.g. Korte and Vygen, 2005). More precisely, consider the quality of the projects as their value and the required funding as their weight. Then the size of the knapsack is given by the budget and we have a knapsack problem where the items are controlled by players who choose the weight of the item strategically.[1]

[1] Aggarwal and Hartline (2006) and Dizdar et al. (2011) also analyse incomplete information variants of the knapsack problem. In their papers, a profit maximizing seller wants

Finally, we should mention that this problem can be seen as the Utility Maximization problem with private information.

1. THE MODEL

Assume that there are N potential sellers and let $i \in \{1, \dots, N\}$ denote a typical seller. Each seller has an item to sell for which only he knows his private reserve price, θ_i, i.e., $\theta_i \in [\underline{\theta_i}, \overline{\theta_i}]$ with $0 \leq \underline{\theta_i} < \overline{\theta_i} < \infty$ for all i.
Denote $\theta := (\theta_1, \dots, \theta_N)$ and $\Theta := [\underline{\theta}_1, \overline{\theta}_1] \times \dots \times [\underline{\theta}_N, \overline{\theta}_N]$. As usual, the subscript $-i$ denotes a vector with the ith component removed (or a product with the ith factor removed).
We shall impose the classic assumption that there exist probability density functions $f_i : [\underline{\theta}_i, \overline{\theta}_i] \to \mathbb{R}$ for i's reservation price, θ_i, which is common knowledge. We assume that the f_i are continuous and strictly positive functions on $[\underline{\theta}_i, \overline{\theta}_i]$. Furthermore, let $F_i : [\underline{\theta}_i, \overline{\theta}_i] \to [0, 1]$ be the corresponding cumulative distribution functions such that

$$F_i(\theta_i) = \int_{\underline{\theta}_i}^{\theta_i} f_i(s_i) ds_i. \tag{1}$$

We also assume that the distributions of the θ_i are independent random variables. We denote the joint densities and cumulative distribution functions by f and F, respectively.
Items are differentiated by a fixed quality $w_i > 0$, which are common knowledge. While this might not always be a perfect model of real world szenarios, we believe that it still poses a good approximation, in particular for cases where the same circle of sellers apply repeatedly. We thereby assume that the differences in qualities can be expressed qualitatively, i.e. that the experts can state cardinal preferences for the individual projects.
Sellers are assumed to be risk–neutral. A seller with reserve price θ_i who receives a price t_i in return for selling his item with probability q_i has utility

$$u_i = t_i - \theta_i q_i. \tag{2}$$

Sellers are faced by a single buyer who has a finite budget B to acquire as many items as possible weighted by quality. In particular, money does not enter the buyer's objective function, it only enters his (budget) constraint. Thus, the buyer's problem is to find a mechanism that maximizes $E_\Theta(\sum_{i=1}^N w_i q_i(\theta))$ subject to sellers' incentive and participation constraints and his own budget constraint.[2]

to sell a given capacity to a set of buyers, i.e. the seller has a 'classic' objective function.

[2]The particular nature of the objective function deserves a caution note. An additive function like this assumes that projects are substitutes. This might not always be the case in reality. It could well be the case that there are two very similar projects of which an organization would only like to fund one.

More formally, let C_N denote the N–dimensional unit cube,

$$C_N = \{ (x_1, \dots, x_N) | \, 0 \le x_i \le 1 \} \, . \tag{3}$$

Define a direct mechanism as a pair of functions q and t where

$$q : \Theta \to C_N, \tag{4}$$

$$t : \Theta \to \mathbb{R}^N. \tag{5}$$

Let $q_i(\theta)$ and $t_i(\theta)$ denote the ith components of q and t, respectively.
For now we shall only impose a soft budget constraint, i.e. we will merely require that the budget is not exceeded in expectation. We later show that this is no restriction.
So the buyer's problem is

$$\max_{\{q,t\}} \int_\Theta \sum_{i=1}^N w_i q_i(\theta) f(\theta) d(\theta) \tag{6}$$

$$\text{s.t. } U_i(\theta_i|\theta_i) \ge 0 \; \forall i, \theta_i, \tag{7}$$

$$U_i(\theta_i|\theta_i) \ge U_i(\theta_i'|\theta_i) \; \forall i, \theta_i, \theta_i', \tag{8}$$

$$\int_\Theta \sum_{i=1}^N t_i(\theta) f(\theta) d\theta \le \mathcal{B}, \tag{9}$$

where

$$U_i(\tilde{\theta}_i|\theta_i) := \int_{\Theta_{-i}} \Big(t_i(\tilde{\theta}_i, \theta_{-i}) - \theta_i q_i(\tilde{\theta}_i, \theta_{-i}) \Big) f_{-i}(\theta_{-i}) d\theta_{-i} \tag{10}$$

denotes the expected utility of a seller with type θ_i if he reports type $\tilde{\theta}_i$ and all other sellers report truthfully.
We shall denote

$$U_i(\theta_i) := U_i(\theta_i|\theta_i). \tag{11}$$

Similarly, define

$$T_i(\theta_i) := \int_{\Theta_{-i}} t_i(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i}) d\theta_{-i}, \tag{12}$$

$$Q_i(\theta_i) := \int_{\Theta_{-i}} q_i(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i}) d\theta_{-i} \tag{13}$$

to be the expected transfer and the expected probability of selling for seller i, respectively.
The buyer's expected utility from seller i is

$$w_i \int_{\underline{\theta_i}}^{\overline{\theta_i}} Q_i(\theta_i) f_i(\theta_i) d\theta_i. \tag{14}$$

Thus, the buyer's problem is

$$\max_{\{q,t\}} \sum_{i=1}^{N} w_i \int_{\underline{\theta_i}}^{\overline{\theta_i}} Q_i(\theta_i) f_i(\theta_i) d\theta_i \tag{15}$$

$$\text{s.t.} \quad T_i(\theta_i) - \theta_i Q_i(\theta_i) \geq 0 \; \forall i, \theta_i, \tag{16}$$

$$T_i(\theta_i) - \theta_i Q_i(\theta_i) \geq T_i(\tilde{\theta}_i) - \theta_i Q_i(\tilde{\theta}_i) \; \forall i, \theta_i, \tilde{\theta}_i, \tag{17}$$

$$\int_{\Theta} \sum_{i=1}^{N} t_i(\theta) f(\theta) d\theta \leq \mathcal{B}. \tag{18}$$

2. ANALYSIS

The technical proofs of this section are left to the appendix.
Standard analysis shows that

Lemma 1. *The two conditions*

$$U_i(\theta_i|\theta_i) \geq 0, \quad \forall i, \theta_i \tag{19}$$

$$U_i(\theta_i|\theta_i) \geq U_i(\tilde{\theta}_i|\theta_i), \quad \forall i, \theta_i, \tilde{\theta}_i, \theta_i < \tilde{\theta}_i \tag{20}$$

are equivalent to

$$Q_i(\theta_i) \geq Q_i(\tilde{\theta}_i), \quad \forall i, \theta_i, \tilde{\theta}_i, \theta_i < \tilde{\theta}_i \tag{21}$$

$$U_i(\theta_i|\theta_i) = U_i(\overline{\theta}_i|\overline{\theta}_i) + \int_{\theta_i}^{\overline{\theta}_i} Q_i(s) ds, \quad \forall i, \theta_i \tag{22}$$

$$U_i(\overline{\theta}_i|\overline{\theta}_i) \geq 0. \tag{23}$$

Equations (22) are equivalent to

$$T_i(\theta_i) = T_i(\overline{\theta}_i) - \overline{\theta}_i Q_i(\overline{\theta}_i) + \theta_i Q_i(\theta_i) + \int_{\theta_i}^{\overline{\theta}_i} Q_i(s) ds. \tag{24}$$

In the following, we refer to a *feasible mechanism* as any mechanism that satisfies (18) and (21)-(23).
Next, we show that the expected utility of the worst type can be set to zero.

Lemma 2. *W.l.o.g. we can restrict attention to mechanisms satisfying* $U_i(\overline{\theta}_i|\overline{\theta}_i) = 0$ *for all* $i = 1, \ldots, N$.

Then (24) becomes

$$T_i(\theta_i) = \theta_i Q_i(\theta_i) + \int_{\theta_i}^{\overline{\theta}_i} Q_i(s) ds. \tag{25}$$

The budget constraint, (18), can be written as

$$\sum_{i=1}^{N} \int_{\underline{\theta}_i}^{\overline{\theta}_i} T_i(\theta_i) f_i(\theta_i) d\theta_i \leq \mathcal{B}. \tag{26}$$

Clearly, with an arbitrarily large budget the whole problem is trivial. More precisely, if $\mathcal{B} \geq \sum_{i=1}^{N} \overline{\theta_i}$, then the mechanism $q_i(\theta) = 1$ and $t_i(\theta) = \overline{\theta}_i$ for all i is trivially optimal.
Thus, in the following we assume that $\mathcal{B} < \sum_{i=1}^{N} \overline{\theta_i}$. It follows that

Lemma 3. *The budget constraint,* (26)*, is binding.*

Applying Lemma 3 and inserting (25) in (26) gives

$$\sum_{i=1}^{N} \int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\theta_i Q_i(\theta_i) + \int_{\theta_i}^{\overline{\theta_i}} Q_i(s) ds \right) f_i(\theta_i) d\theta_i = \mathcal{B} \tag{27}$$

for any optimal mechanism. Integrating by parts, this gives

$$\sum_{i=1}^{N} \int_{\underline{\theta_i}}^{\overline{\theta_i}} Q_i(\theta_i) \left(\frac{F_i(\theta_i)}{f_i(\theta_i)} + \theta_i \right) f_i(\theta_i) d\theta_i = \mathcal{B}, \tag{28}$$

where each summand is the buyer's *ex ante* expected payment to seller i.
Then,

$$\int_{\underline{\theta_i}}^{\overline{\theta_i}} Q_i(\theta_i) \left(\frac{F_i(\theta_i)}{f_i(\theta_i)} + \theta_i \right) f_i(\theta_i) d\theta_i = \underbrace{\mathcal{B} - \sum_{j \neq i} \int_{\underline{\theta_j}}^{\overline{\theta_j}} Q_j(\theta_j) \left[\frac{F_j(\theta_j)}{f_j(\theta_j)} + \theta_j \right] f_j(\theta_j) d\theta_j}_{=:\mathcal{B}_i} . \tag{29}$$

The next lemma shows that, in the optimal mechanism, a seller's expected transfer can never exceed that seller's highest type.

Lemma 4. $\mathcal{B}_i \leq \overline{\theta_i}$.

Let $Q := (Q_1, \ldots, Q_N)$ and $T := (T_1, \ldots, T_N)$ be functions satisfying (21), (25), and (28). Then note that setting $q_i(\theta) := Q_i(\theta_i)$ and $t_i(\theta) := T_i(\theta_i)$ defines a feasible mechanism.
Hence, there exists an optimal mechanism where each seller's ex-post allocation equals its interim allocation.
This is remarkable, as it implies that there exists an optimal mechanism in which there is no 'ex-post competition', i.e. seller i's allocation depends only on her own report and all the priors (supports and distributions) but not on other sellers' types.

Let $\overline{\mathcal{B}} := \left\{(\mathcal{B}_1, \dots, \mathcal{B}_N) | \mathcal{B}_i \in [0, \overline{\theta_i}]\right\}$. Then our problem is

$$\max_{\overline{\mathcal{B}}} \sum_{i=1}^{N} P_i(\mathcal{B}_i) \ \text{ s.t. } \sum_{i=1}^{N} \mathcal{B}_i = \mathcal{B}, \tag{30}$$

where

$$P_i(\mathcal{B}_i) := \max_{Q_i} \int_{\underline{\theta_i}}^{\overline{\theta_i}} w_i Q_i(\theta_i) f_i(\theta_i) d\theta_i \tag{31}$$

$$\text{s.t.} \quad Q_i(\theta_i) \geq Q_i(\theta_i') \text{ if } \theta_i \leq \theta_i', \tag{32}$$

$$\int_{\underline{\theta_i}}^{\overline{\theta_i}} Q_i(\theta_i) \left[\frac{F_i(\theta_i)}{f_i(\theta_i)} + \theta_i\right] f_i(\theta_i) d\theta_i = \mathcal{B}_i. \tag{33}$$

Lemma 5. *The set of $Q_i(\theta_i)$ that satisfy* (32) *and* (33) *is not empty.*

This establishes

Observation 1. *Ex-post competition is not necessary for optimality.*

Let us see if we can characterize in more detail. For this, define player i's *virtual cost* function,

$$v_i(\theta_i) := \frac{F_i(\theta_i)}{f_i(\theta_i)} + \theta_i. \tag{34}$$

Then

Lemma 6 (Neyman-Pearson)**.** *Ignoring the monotonicity requirement,* (32)*, an optimal function Q_i^* is given by*

$$Q_i^*(\theta_i) = \begin{cases} 1 & \text{if } v_i(x) \leq c \\ 0 & \text{if } v_i(x) > c, \end{cases} \tag{35}$$

where $c \geq 0$ is chosen such that (33) *is satisfied.*

Generally, function Q_i^* from Lemma 6 is not monotone, as required in (32). We can however, specify an optimal function Q_i for a subset of distributions.

Corollary 1. *If F_i is regular, i.e., $v(x)$ is strictly increasing, then there exists $\theta_i^* \in [\underline{\theta}_i, \overline{\theta}_i]$ such that the step function*

$$Q^*(\theta_i) := \begin{cases} 1 & \text{if } \theta_i \leq \theta_i^* \\ 0 & \text{if } \theta_i > \theta_i^* \end{cases} \tag{36}$$

is optimal. Thus, in this case it is optimal to make multiple simultaneous take-it-or-leave-it offers.

From now on, assume that F_i is regular, i.e., v_i is strictly increasing.
But then,

$$\max_{Q_i} \int_{\underline{\theta_i}}^{\overline{\theta_i}} w_i Q_i(\theta_i) f_i(\theta_i) d\theta_i = \max_{\theta_i^*} \int_{\underline{\theta_i}}^{\theta_i^*} w_i f_i(\theta_i) d\theta_i = \max_{\theta_i^*} w_i F(\theta_i^*), \quad (37)$$

and (33) becomes

$$\int_{\underline{\theta_i}}^{\theta_i^*} F_i(\theta_i) d\theta_i + \int_{\underline{\theta_i}}^{\theta_i^*} \theta_i f_i(\theta_i) d\theta_i = \mathcal{B}_i \quad (38)$$

Integrating $F_i(\theta_i)$ by parts, we get that (33) becomes

$$\theta_i^* F_i(\theta_i^*) = \mathcal{B}_i. \quad (39)$$

But since $\theta_i^* F_i(\theta_i^*)$ is a strictly increasing function, the max operator in (37) is redundant and we get

$$P_i(\mathcal{B}_i) = w_i F_i(\theta_i^*) \text{ s.t. } \theta_i^* F_i(\theta_i^*) = \mathcal{B}_i. \quad (40)$$

Summing up, our mechanism design problem is now

$$\max_{\Theta} \sum_{i=1}^{N} w_i F_i(\theta_i^*) \text{ s.t. } \sum_{i=1}^{N} \theta_i^* F_i(\theta_i^*) = \mathcal{B}. \quad (41)$$

Summarizing the above, we arrive at the following.

Optimal mechanism There exists θ_i^* for all i s.t. $\sum_{i=1}^N \theta_i^* F_i(\theta_i^*) = \mathcal{B}$ so that

$$q_i(\theta) = \begin{cases} 1 & \text{if } \theta_i \leq \theta_i^* \\ 0 & \text{else.} \end{cases} \quad (42)$$

$$T_i(\theta_i) = \theta_i Q_i(\theta_i) + \int_{\underline{\theta_i}}^{\theta_i} Q_i(s) ds. \quad (43)$$

We can set $t_i(\theta) := T_i(\theta_i)$ for all i. This mechanism is

1. Ex–post IC
2. Ex–post IR
3. Interim budget–feasible.

However, we can use AGV–budget–balancing (see, e.g., Börgers and Norman, 2009) to get a mechanism which is ex–post budget–feasible. For this, set

$$t_i(\theta) := T_i(\theta_i) - \frac{1}{N-1} \sum_{j \neq i} T_j(\theta_j) + \int_{\theta_{-i}} \frac{1}{N-1} \sum_{j \neq i} T_j(\theta_j) d\theta_{-i}. \quad (44)$$

This mechanism is

1. Ex–post IC
2. Ex–post budget–feasible
3. intermim IR.

Note that it is *not* ex–post IR in general.
We show that the problem is characterized by FOCs. To this end, we need

Lemma 7. *$P_i(\mathcal{B}_i)$ is a concave function.*

Using Lemma 7 we can characterize the solution of program (40). Using the Lagrange multiplier λ for the budget constraint, and μ_i for the constraints $\theta_i \geq \underline{\theta}_i$ and γ_i for $\theta_i \leq \overline{\theta}_i$, we get the first-order conditions

$$w_i f_i(\theta_i) = \lambda(F_i(\theta_i) + \theta_i f_i(\theta_i)) - \mu_i + \gamma_i \qquad \forall i = 1, \ldots, N \tag{45}$$

$$\mu_i(\underline{\theta}_i - \theta_i) = 0, \quad \theta_i \geq \underline{\theta}_i, \qquad \forall i = 1, \ldots, N \tag{46}$$

$$\gamma_i(\theta_i - \overline{\theta}_i) = 0, \quad \theta_i \leq \overline{\theta}_i, \qquad \forall i = 1, \ldots, N \tag{47}$$

$$\sum_{i=1}^{N} \theta_i F_i(\theta_i) = \mathcal{B}. \tag{48}$$

Conditions (45) - (47) reveal that, in principle, the optimal mechanism can treat players in three different ways. Player i's cutoff value θ_i^* might be in the range $(\underline{\theta}_i, \overline{\theta}_i)$, i.e. he might have strictly positive selling probability (but does not sell with certainty). Thus, $\mu_i = \gamma_i = 0$, by (46) and (47). Then there might be players l with zero selling probability, i.e., $\mu_l > 0$ and $\theta_l^* = \underline{\theta}_l$. Finally, there might be sellers k who will sell with certainty, $\gamma_k > 0$ and $\theta_k^* = \overline{\theta}_k$. However, by (45), for all these players i, l and k, the ratios below must be equal (to λ).

$$\lambda = \frac{w_i f_i(\theta_i)}{F_i(\theta_i) + \theta_i f_i(\theta_i)} = \frac{w_l f_l(\underline{\theta}_l) + \mu_l}{F_l(\underline{\theta}_l) + \underline{\theta}_l f_l(\underline{\theta}_l)} = \frac{w_k f_k(\overline{\theta}_k) - \gamma_k}{F_k(\overline{\theta}_k) + \overline{\theta}_k f_k(\overline{\theta}_k)}, \quad \mu_j, \gamma_k > 0 \tag{49}$$

Thus, all players i and j who sell with strictly positive probability (but not with certainty) must have the same ratio of quality and virtual cost (just writing the above condition for player i in terms of $v_i(\theta_i)$ and simplifying) and that ratio is equal to the multiplier for the (binding) budget constraint.

$$\lambda = \frac{w_i}{v_i(\theta_i)} = \frac{w_j}{v_j(\theta_j)}, \quad \text{for all } i, j \text{ with } \theta_i^* \in (\underline{\theta}_i, \overline{\theta}_i), \theta_j^* \in (\underline{\theta}_j, \overline{\theta}_j) \tag{50}$$

Lemma 8. *The optimal mechanism has the following properties:*

1. *$\frac{w_i}{v_i(\theta_i^*)} = \frac{w_j}{v_j(\theta_j^*)}$ for all i, j where the optimal mechanism prescribes $\underline{\theta}_i < \theta_i^* < \overline{\theta}_i$ and $\underline{\theta}_i < \theta_i^* < \overline{\theta}_i$.*

2. *If $\underline{\theta}_i = 0$ for some player i, then $\theta_i^* > \underline{\theta}_i$, i.e., player i is in the allocation with strictly positive probability.*

3. *In case of symmetry, i.e., all players are ex ante equal, $w_1 = \cdots = w_N$, $F_1 = \cdots = F_N$, all players have the same strictly positive selling probability, but they do not sell with certainty.*[3]

Example Let $N = 2$, $F_i = U[0,1]$, $w_1 = w_2$, $\mathcal{B} = 1$. Then the problem is $\max_\theta \theta_1 + \theta_2$ s.t. $\theta_1^2 + \theta_2^2 = 1$.

$$L(\lambda) = \theta_1 + \theta_2 - \lambda(\theta_1^2 + \theta_2^2 - 1) \tag{51}$$

$$\Rightarrow 1 = 2\lambda\theta_1 \;\Rightarrow\; \theta_1 = \frac{1}{2\lambda} \tag{52}$$

$$1 = 2\lambda\theta_2 \;\Rightarrow\; \theta_2 = \frac{1}{2\lambda}. \tag{53}$$

Now choose λ s.t. $\theta_1^2 + \theta_2^2 = 1 = \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2$.

$$\Rightarrow \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = \frac{1}{2\lambda^2} = 1 \tag{54}$$

$$\Rightarrow \lambda = \frac{1}{\sqrt{2}} \tag{55}$$

$$\Rightarrow \theta_1^* = \theta_2^* = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}. \tag{56}$$

So,

$$q_1 = q_2 = \begin{cases} 1 & \text{if } \theta_i \leq \frac{1}{\sqrt{2}} \\ 0 & \text{else.} \end{cases} \tag{57}$$

$$T_i(\theta_i) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } \theta_i \leq \frac{1}{\sqrt{2}} \\ 0 & \text{else.} \end{cases} \tag{58}$$

If we want ex–post budget–feasibility,

$$t_i(\theta) = T_i(\theta_i) - T_j(\theta_j) + \int_0^1 T_j(\theta_j) d\theta_j \tag{59}$$

$$= T_i(\theta_i) - T_j(\theta_j) + \frac{1}{2}. \tag{60}$$

[3](excluding the non-generic case where $\theta_i^* = \overline{\theta}$ is an interior solution of the problem if one ignores the constraint $\theta_i \leq \overline{\theta}$).

3. A DETAIL-FREE MECHANISM

In the last section we derived a (procurer) optimal Bayesian mechanism. This model assumes that all distributions are public knowledge. This is a good assumption in many settings, e.g. in settings where the same sellers compete again and again. However, in other settings the assumption might be too restrictive. What can we say if the distributions are not known?
Clearly, there is no mechanism that is globally optimal in the sense that it is at least as good as any other mechanism on any instance. Thus, there is no 'detail free' optimal mechanism. But if we restrict the class of mechanisms in a suitable way we can find such a mechanism. In this section, we will present a simple and intuitive auction mechanism that pays a uniform transfer relative to the quality of the item. In the special case where all qualities are the same, this mechanism is detail free optimal among all mechanisms that pay all sellers a uniform transfer. We introduced this mechanism in Ensthaler and Giebe (2009). Papadimitriou and Singer (2010) analyse the worst case performance of a similar mechanism. The mechanism can be seen as an extension of the greedy split heuristic to this game theoretic knapsack problem.

3.1. *The detail-free setting*

The buyer's objective is now to purchase a subset of items of maximal quality, where the quality of a set of items is the sum of qualities of its elements. As before, money does not enter the buyer's utility function. We assume that the probability of two types being equal is zero and that there is no known statistical relation between sellers' types, resp. between quality and types.
We consider an open auction mechanism, where the sellers submit bids $b_i \geq 0$. The vector of sellers' bids is $b = (b_i, b_{-i}) = (b_1, \ldots, b_N)$. Let $\mathcal{A}(b) := \{i \in \mathcal{I} | q_i(b) = 1\}$ denote the *allocation* resulting from the play of the auction when the vector of final bids is b, and denote the complementary set by $\mathcal{A}^c$. The allocation is the set of sellers whose objects are procured. For simplicity, denote $u_i(b) := u_i(b|\theta_i)$ and $\mathcal{A} := \mathcal{A}(b)$ if confusion is unlikely.
The buyer's utility is the total quality of the procured items,

$$u_{\text{buyer}}(\mathcal{A}) = \sum_{j \in \mathcal{A}} w_j = \sum_{i \in \mathcal{I}} q_i w_i, \quad q_i \in \{0, 1\}. \tag{61}$$

Under complete information, the buyer faces a simple knapsack problem,

$$\arg\max_{q} \sum_{i \in \mathcal{I}} w_i q_i \ s.t. \sum_{i \in \mathcal{I}} \theta_i q_i \leq \mathcal{B}, \ q \in \{0, 1\}^N. \tag{62}$$

Call (62) the *first-best* allocation(s).

3.2. *The Auction*

Consider the following open descending clock auction. A central continuous clock counts down the price-per-quality ratio r, beginning from the *highest* initial ratio among all bidders, $r^{\max} := \max_{i \in \mathcal{I}} \mathcal{B}/w_i$. A bidder can quit the auction at any time, but can never come back. Being active at clock reading r implies that bidder i is willing to accept a payment of rw_i in return for his item.

As the clock counts down, each active bidder's current financial bid, rw_i, decreases. As long as the sum of active financial bids exceeds the budget, the countdown continues. As soon as that sum fits in the budget, say, at round r^f, the auction ends, and each *active* bidder i sells his item at the price $r^f w_i$.

The *sum* of active financial bids decreases smoothly as the clock counts down, until some bidder j exits, at which point that sum is reduced by the amount rw_j.

Two events can end the auction. First, the end can be triggered by an exit, when before the exit, the sum of active financial bids exceeded the budget, but after the exit the remaining active bids can be accomodated without exhausting the budget. Second, it may end because at some point all active bids exactly use up the budget.

After the auction, each loser j has a *final* ratio $r_j = b_j/w_j$ with financial bid b_j since j quit the auction at clock reading $r = r_j$. All winners i have the same final ratio $r^f = b_i/w_i$ with (different) selling prices b_i.

In order to give a direct characterization of the auctions' equilibrium result we need the following notation.

Let l_i denote the set of players with final ratio not above that of bidder i. Let $r_{\mathcal{A}^c}$ denote the *smallest* ratio among all players who are *not* in the allocation (unless $\mathcal{A}^c = \emptyset$).

$$l_i := \left\{ j \in \mathcal{I} \,\middle|\, \frac{b_j}{w_j} \leq \frac{b_i}{w_i} \right\} \tag{63}$$

$$r_{\mathcal{A}^c} := \begin{cases} \min\left\{ \frac{b_j}{w_j} \,\middle|\, j \in \mathcal{A}^c \right\} & \text{if } \mathcal{A}^c \neq \emptyset \\ \frac{\mathcal{B}}{\min_{j \in \mathcal{I}} w_j} & \text{if } \mathcal{A}^c = \emptyset \end{cases} \tag{64}$$

Lemma 9.

1. *The stop rule $b_i(\theta_i) = \theta_i$ is a weakly dominant strategy.*

2. *In equilibrium (setting $b_i = \theta_i$), the auction implements allocation $\mathcal{A}$*

and payments t where

$$q_i(b) = \begin{cases} 1 & \text{if } \frac{b_i}{w_i}\sum_{j\in l_i} w_j \le \mathcal{B} \\ 0 & \text{otherwise}, \end{cases} \tag{65}$$

$$t_i(b) = \begin{cases} \min\left\{ r_{\mathcal{A}^c} w_i, \frac{\mathcal{B}}{\sum_{j\in\mathcal{A}} w_j} w_i \right\} & \text{if } q_i(b) = 1 \\ 0 & \text{otherwise}. \end{cases} \tag{66}$$

3. *Truth-telling is ex-post rational.*

4. *The mechanism is ex-post budget feasible.*

Let us now show that, if all bidders are ex ante identical, the auction mechanism is detail-free optimal among all mechanisms that are deterministic and pay all sellers in the allocation a uniform price per unit of quality.

Lemma 10. *Let $\theta \in \Theta$, $w_i = w_j$ for all i, j and let $(\tilde{q}, \tilde{t})$ be any feasible, deterministic mechanism with $\tilde{t}_i(\theta) = \tilde{t}_j(\theta)$ for all i, j with $\tilde{q}_i(\theta) = \tilde{q}_j(\theta) = 1$. Then $\sum_{k=1}^N q_i(\theta) \ge \sum_{k=1}^N \tilde{q}_i(\theta)$.*

Finally, we should mentioned there is another meaningful measure of optimality in the detail free setting that we could have chosen for our analysis, namely that worst-case performance of a mechanism. For the special case where all items have the same quality, one can easily show that the auction is a 2-approximation, which means that in every instance, the first-best allocation is at most twice as good as the allocation chosen by the auction:[4]

Lemma 11 (2-approx of First-Best)**.** *Suppose $w_i = w_j$ for all i and j. The proposed auction mechanism is a* 2-approximation *mechanism of the first-best solution and this approximation is* tight*, i.e., if the auction selects an allocation containing c items, then the first-best allocation does not contain more than* $2c$ *items,* $c \in \mathbb{N}$, $c > 0$.

4. CONCLUSION

We derived the Bayesian optimal mechanism for a novel type of mechanism design problem, namely the Bayesian optimal budget feasible mechanism for an additive objective function. We show that the mechanism can be implemented in dominant strategies and does not require ex-post competition. We use this mechanism to explain the granting procedure of public agencies such as the National Science Foundation. We see room for further research. For example, what can we say about the optimal selling mechanisms for

[4]For the general case with different qualities Papadimitriou and Singer (2010) show that a slightly modified direct mechanism is a 6-approximation.

more general, non-additive objective functions. Also, it would be interesting to derive the optimal ex-post mechanism, i.e. the optimal mechanism in the class of mechanisms which are Dominant strategy incentive compatible, ex-post individually rational and observe the ex-post budget constraint.

5. APPENDIX

Proof of Lemma 1. First, we show that (19) and (20) imply (21), (22), and (23). Start with (20):

$$\begin{aligned} U_i(\theta_i|\theta_i) &\geq U_i(\tilde{\theta}_i|\theta_i) && (67)\\ &= T_i(\tilde{\theta}_i) - \theta_i Q_i(\tilde{\theta}_i) && (68)\\ &= T_i(\tilde{\theta}_i) - \tilde{\theta}_i Q_i(\tilde{\theta}_i) + \tilde{\theta}_i Q_i(\tilde{\theta}_i) - \theta_i Q_i(\tilde{\theta}_i) && (69)\\ &= U_i(\tilde{\theta}_i|\tilde{\theta}_i) + Q_i(\tilde{\theta}_i)(\tilde{\theta}_i - \theta_i). && (70) \end{aligned}$$

Analogously, (with the roles of θ_i and $\tilde{\theta}_i$ interchanged)

$$U_i(\tilde{\theta}_i|\tilde{\theta}_i) \geq U_i(\theta_i|\theta_i) + Q_i(\theta_i)(\theta_i - \tilde{\theta}_i). \quad (71)$$

Suppose $\theta_i < \tilde{\theta}_i$. Then we can combine (70) and (71) as

$$Q_i(\tilde{\theta}_i) \leq -\frac{U_i(\theta_i|\theta_i) - U_i(\tilde{\theta}_i)|\tilde{\theta}_i}{\theta_i - \tilde{\theta}_i} \leq Q_i(\theta_i), \quad (72)$$

which implies (21). From (72) we get (analogous to Myerson (1981))

$$U_i'(\theta_i|\theta_i) = -Q_i(\theta_i), \quad (73)$$

which, after integrating, becomes (22). Finally, (19) together with (22) implies (23).
Now prove that (21), (22), and (23) imply (19) and (20). Suppose $\theta_i < \tilde{\theta}_i$. We have

$$\begin{aligned} U_i(\theta_i|\theta_i) - U_i(\tilde{\theta}_i|\tilde{\theta}_i) &= \int_{\theta_i}^{\tilde{\theta}_i} Q_i(s)ds && (74)\\ &\geq Q_i(\tilde{\theta}_i)(\tilde{\theta}_i - \theta_i), && (75) \end{aligned}$$

where the first line follows from (22), and the second from (21). This, however, implies (20):

$$\begin{aligned} U_i(\theta_i|\theta_i) &\geq U_i(\tilde{\theta}_i|\tilde{\theta}_i) + Q_i(\tilde{\theta}_i)(\tilde{\theta}_i - \theta_i) && (76)\\ &= T_i(\tilde{\theta}_i) - \tilde{\theta}_i Q_i(\tilde{\theta}_i) + Q_i(\tilde{\theta}_i)(\tilde{\theta}_i - \theta_i) && (77)\\ &= T_i(\tilde{\theta}_i) - \theta_i Q_i(\tilde{\theta}_i) && (78)\\ &= U_i(\tilde{\theta}_i|\theta_i) && (79) \end{aligned}$$

Finally, (22) and (23) together imply (19).

□

Proof of Lemma 2. Take any mechanism (q, t) with corresponding $T_i(\theta_i)$ and $Q_i(\theta_i)$ that satisfies (18) and (21)-(23) and suppose there is a player i for which $U_i(\overline{\theta}_i|\overline{\theta}_i) = T_i(\overline{\theta}_i) - \overline{\theta}_i Q_i(\overline{\theta}_i) > 0$. Then the mechanism (q, t') where $t'_i = t_i - U_i(\overline{\theta}_i|\overline{\theta}_i)$ is a feasible mechanism where $U'_i(\overline{\theta}_i|\overline{\theta}_i) = 0$. Thus, any value of the objective function, (15), that is achievable with $U_i(\overline{\theta}_i|\overline{\theta}_i) > 0$ is also achievable with $U_i(\overline{\theta}_i|\overline{\theta}_i) = 0$. □

Proof of Lemma 3. The proof is done in two steps. First, we show that for any feasible mechanism (q, t) there exist i and $\tilde{\theta}_i < \overline{\theta}_i$ such that $Q_i(\theta_i) < 1$ for all $\theta_i \geq \tilde{\theta}_i$. Suppose not. Then $Q_i(\theta_i) = 1$ almost everywhere for all i. But then, by (25), $\int_{\underline{\theta}_i}^{\overline{\theta}_i} T_i(\theta_i) f_i(\theta_i) d\theta_i = \overline{\theta}_i$ for all i. By feasibility, we know that $\sum_{i=1}^{N} \int_{\underline{\theta}_i}^{\overline{\theta}_i} T_i(\theta_i) f_i(\theta_i) d\theta_i \leq \mathcal{B}$. But then, $\sum_{i=1}^{N} \overline{\theta}_i \leq \mathcal{B}$, a contradiction to our assumption that $\mathcal{B} < \sum_{i=1}^{N} \overline{\theta}_i$.
Second, we show that a mechanism with $\sum_{i=1}^{N} \int_{\underline{\theta}_i}^{\overline{\theta}_i} T_i(\theta_i) f_i(\theta_i) d\theta_i < \mathcal{B}$ is not optimal since the value of the objective, (15), can be improved without exceeding the budget.
In order to see this, take any feasible mechanism (q, t) where the budget is not exhausted, $\sum_{i=1}^{N} \int_{\underline{\theta}_i}^{\overline{\theta}_i} T_i(\theta_i) f_i(\theta_i) d\theta_i < \mathcal{B}$. Let θ_i^* be the maximal θ_i such that $Q_i(\theta_i) = 1$, and if it does not exist, let θ_i^* be the minimal θ_i such that $Q_i(\theta_i) < 1$ (by the above, we know such a θ_i^* exists). Let $\Delta > 0$ and define a mechanism (q', t') where

$$q'_i(\theta_i, \theta_{-i}) = \begin{cases} 1 & \theta_i \in [\underline{\theta}_i, \theta_i^* + \Delta] \\ q_i(\theta_i, \theta_{-i}) & \theta_i > \theta_i^* + \Delta \end{cases} \tag{80}$$

$$t'_i(\theta_i, \theta_{-i}) = \begin{cases} \theta_i^* + \Delta + \int_{\theta_i^*+\Delta}^{\overline{\theta}_i} Q_i(s) ds & \theta_i \in [\underline{\theta}_i, \theta_i^* + \Delta] \\ t_i(\theta_i, \theta_{-i}) & \theta_i > \theta_i^* + \Delta. \end{cases} \tag{81}$$

For this mechanism, it is straightforward to show that

$$Q'_i(\theta_i) = \begin{cases} 1 & \theta_i \in [\underline{\theta}_i, \theta_i^* + \Delta] \\ Q_i(\theta_i) & \theta_i > \theta_i^* + \Delta \end{cases}, \tag{82}$$

$$T'_i(\theta_i) = \begin{cases} t'_i(\theta_i, \theta_{-i}) & \theta_i \in [\underline{\theta}_i, \theta_i^* + \Delta] \\ T_i(\theta_i) & \theta_i > \theta_i^* + \Delta. \end{cases} \tag{83}$$

This mechanism is feasible: First, (25) and (21) are satisfied (the former is easily shown and the latter is obvious). Second, if $U_i(\overline{\theta}_i|\overline{\theta}_i) = 0$ is satisfied under (q, t) then it is also satisfied under (q', t'). Third, since $Q'_i(\theta_i)$ is monotonic and thus continuous almost everywhere, it follows that there is a sufficiently small Δ such that (q', t') satisfies the budget constraint, (26). Since $\int_{\underline{\theta}_i}^{\overline{\theta}_i} Q'_i(\theta_i) f_i(\theta_i) d\theta_i > \int_{\underline{\theta}_i}^{\overline{\theta}_i} Q_i(\theta_i) f_i(\theta_i) d\theta_i$, the value of the objective function, (15), of (q', t') exceeds that of (q, t). □

Proof of Lemma 4.

$$
\begin{aligned}
B_i &= \int_{\underline{\theta_i}}^{\overline{\theta_i}} Q_i(\theta_i) \left(\frac{F_i(\theta_i)}{f_i(\theta_i)} + \theta_i \right) f_i(\theta_i) d\theta_i && (84) \\
&= \int_{\underline{\theta_i}}^{\overline{\theta_i}} Q_i(\theta_i) F_i(\theta_i) d\theta_i + \int_{\underline{\theta_i}}^{\overline{\theta_i}} Q_i(\theta_i)\theta_i f_i(\theta_i) d\theta_i && (85) \\
&\leq \int_{\underline{\theta_i}}^{\overline{\theta_i}} F_i(\theta_i) d\theta_i + \int_{\underline{\theta_i}}^{\overline{\theta_i}} \theta_i f_i(\theta_i) d\theta_i && (86) \\
&= \int_{\underline{\theta_i}}^{\overline{\theta_i}} F_i(\theta_i) d\theta_i + [\theta_i F_i(\theta_i)]_{\underline{\theta_i}}^{\overline{\theta_i}} - \int_{\underline{\theta_i}}^{\overline{\theta_i}} F_i(\theta_i) d\theta_i && (87) \\
&= \overline{\theta}_i, && (88)
\end{aligned}
$$

where the third line follows from $Q_i(\theta_i) \leq 1$ and the fourth from integration by parts. □

Proof of Lemma 5. Define

$$
Q_i^*(\theta_i) := \begin{cases} 1 & \text{if } \theta_i \leq \theta_i^* \\ 0 & \text{else,} \end{cases} \tag{89}
$$

which satisfies (32). Then we have

$$
\int_{\underline{\theta}_i}^{\overline{\theta}_i} Q_i^*(s) \left(\frac{F_i(s)}{f_i(s)} + s \right) f_i(s) ds = \int_{\underline{\theta}_i}^{\theta_i^*} \left(\frac{F_i(s)}{f_i(s)} + s \right) f_i(s) ds, \tag{90}
$$

which is equal to zero for $\theta_i^* = \underline{\theta}_i$ and equal to $\overline{\theta}_i$ for $\theta_i^* = \overline{\theta}_i$. By Lemma 4, $\mathcal{B}_i \in [0, \overline{\theta}_i]$. Since (90) is continuous (by continuity of F_i and f_i), there must exist a $\theta_i^* \in [\underline{\theta}_i, \overline{\theta}_i]$ such that (90) is equal to $\mathcal{B}_i$. □

Proof of Lemma 6. Consider (35). First, we argue that there exists a $c \geq 0$ such that (33) is satisfied for every $\mathcal{B}_i \in [0, \overline{\theta}_i]$. For $c = 0$, the LHS of (33) is equal to zero: We have $Q^*(\theta_i) = 0$ almost everywhere, see (35). For $c \geq \max v_i(\theta_i)$, we have $Q^*(\theta_i) = 1$ and, thus, the LHS of (33) is equal to $\overline{\theta}_i$ (after integrating by parts). By continuity of $v_i(\theta_i)$ and $f_i(\theta_i)$, it follows that there is a $c \geq 0$ such that the LHS of (33) takes any value from the range $[0, \overline{\theta}_i]$.
Second, take any function Q_i, that satisfies (32) and (33) and has range

[0, 1].

$$
\begin{aligned}
& (Q_i^*(\theta_i) - Q_i(\theta_i))\,(f_i(\theta_i)(c - v_i(\theta_i))) \geq 0 && (91)\\
\Rightarrow & \int_{\underline{\theta}_i}^{\overline{\theta}_i} (Q_i^*(\theta_i) - Q_i(\theta_i))\,(f_i(\theta_i)(c - v_i(\theta_i)))\,d\theta_i \geq 0 && (92)\\
\iff & c\left(\int_{\underline{\theta}_i}^{\overline{\theta}_i} Q_i^*(\theta_i) f_i(\theta_i) d\theta_i - \int_{\underline{\theta}_i}^{\overline{\theta}_i} Q_i(\theta_i) f_i(\theta_i) d\theta_i\right) && (93)\\
& \geq \int_{\underline{\theta}_i}^{\overline{\theta}_i} Q_i^*(\theta_i) v_i(\theta_i) f_i(\theta_i) d\theta_i - \int_{\underline{\theta}_i}^{\overline{\theta}_i} Q_i(\theta_i) v_i(\theta_i) f_i(\theta_i) d\theta_i = 0 && (94)\\
\Rightarrow & \int_{\underline{\theta}_i}^{\overline{\theta}_i} Q_i^*(\theta_i) f_i(\theta_i) d\theta_i \geq \int_{\underline{\theta}_i}^{\overline{\theta}_i} Q_i(\theta_i) f_i(\theta_i) d\theta_i && (95)
\end{aligned}
$$

There, the first line follows from feasibility of Q_i and (35). The equality in the fourth line follows from feasibility, i.e., both Q_i^* and Q_i satisfy (33). Thus, the objective function, (30), reaches a (weakly) larger value with Q_i^*. □

Proof of Lemma 7. The constraint (39) defines a unique θ_i for every given $\mathcal{B}_i$. Thus, we can write θ_i as a function of $\mathcal{B}_i$ and the constraint becomes $\mathcal{B}_i = \theta_i(\mathcal{B}_i) F_i(\theta_i(\mathcal{B}_i))$, the derivative of which is

$$
\begin{aligned}
1 &= \theta_i'(\mathcal{B}_i) F_i(\theta_i(\mathcal{B}_i)) + \theta_i(\mathcal{B}_i) f_i(\theta_i(\mathcal{B}_i)) \theta_i'(\mathcal{B}_i) && (96)\\
\iff \quad \theta_i'(\mathcal{B}_i) &= \frac{1}{F_i(\theta_i(\mathcal{B}_i)) + \theta_i(\mathcal{B}_i) f_i(\theta_i(\mathcal{B}_i))} && (97)\\
\iff \quad f_i(\theta_i(\mathcal{B}_i))\theta_i'(\mathcal{B}_i) &= \frac{1}{\frac{F_i(\theta_i(\mathcal{B}_i))}{f_i(\theta_i(\mathcal{B}_i))} + \theta_i(\mathcal{B}_i)}. && (98)
\end{aligned}
$$

By the regularity assumption, $F_i(\theta_i)/f_i(\theta_i) + \theta_i$ is strictly increasing. Thus, $f(\theta_i(\mathcal{B}_i))\theta_i'(\mathcal{B}_i)$ is strictly decreasing, which makes $P_i'(\mathcal{B}_i) = w f_i(\theta_i(\mathcal{B}_i))\theta_i'(\mathcal{B}_i)$ strictly decreasing as well. Thus, $P_i(\mathcal{B}_i)$ is concave. □

Proof of Lemma 8.

1. Since $\underline{\theta}_i < \underline{\theta}_i^* < \overline{\theta}_i^*$ and $\underline{\theta}_i < \underline{\theta}_i^* < \overline{\theta}_i^*$, we get $\mu_i = \gamma_i = \mu_j = \gamma_j = 0$ by (46) and (47). Therefore, see (50), the assertion follows.

2. This follows from (49) since $\underline{\theta}_l = 0$ implies division by zero.

3. If players are symmetric, then (by strict monotonicity of v)

$$
\frac{wf(\overline{\theta}) - \gamma_i}{v(\overline{\theta})f(\overline{\theta})} < \frac{wf(\overline{\theta})}{v(\overline{\theta})f(\overline{\theta})} < \frac{wf(\theta_j)}{v(\theta_j)f(\theta_j)} < \frac{wf(\underline{\theta})}{v(\underline{\theta})f(\underline{\theta})} < \frac{wf(\underline{\theta}) + \mu_j}{v(\underline{\theta})f(\underline{\theta})}. \tag{99}
$$

Thus, by (49), all players must have either $\theta_i^* = \overline{\theta}$ or $\theta_i^* = \underline{\theta}$ or $\theta_i^* \in (\underline{\theta}, \overline{\theta})$. The first is not feasible by the proof of Lemma 3 and the second cannot be optimal. Thus, $\theta_i^* \in (\underline{\theta}, \overline{\theta})$ for all players. Then, by (50), and the strict monotonicity of v, $\theta_i^* = \theta^* \in (\underline{\theta}, \overline{\theta})$ for all players i.

□

Proof of Lemma 9.

1. Suppose a seller plays the candidate. His profit is either zero, if he has to quit before the auction ends, or it is positive, if the auction ends before he quits. Consider stopping earlier. Either there is no change or he quits but could have made a profit. Consider stopping later. Either there is no change, or he makes a loss if the auction ends after $r = \theta_i$ but before he quits.

2. By Lemma 9, winners' true ratios are below the final ratio, $\theta_i/w_i < r^f = b_i/w_i$ since they were active at the end of the auction. Thus, winners have lower true ratios than losers. All winners are paid according to the same ratio, r^f, i.e. winner i gets $b_i = r^f w_i$. Thus, i is a winner *iff* it is feasible to pay every player with same or lower final ratio according to i's final ratio, see (65).

   Losers receive no payment and keep their items. The payments to the winners $i \in \mathcal{A}$ depend on which event ends the auction. If the end is triggered by an exit, then the exiting player's final ratio determines the payments, $r_{\mathcal{A}^c} w_i$, otherwise payments are $\frac{\mathcal{B}}{\sum_{j \in \mathcal{A}} w_j} w_i$, i.e., they use up the budget.

   The second line in (64) just makes the definition complete but is never payoff-relevant.

3. Obvious.

4. Obvious.

□

Proof of Lemma 10. Suppose w.l.o.g. that the type vector is ordered, $\theta_1 < \theta_2 < \cdots < \theta_N$ and suppose that the auction mechanism selects allocation $\{1, \ldots, c\}$. Suppose per absurdum that there exists a feasible deterministic mechanism $(\tilde{q}, \tilde{t})$ with uniform transfers to winners and $\sum_{k=1}^N q_i(\theta) < \sum_{k=1}^N \tilde{q}_i(\theta)$. Then there exists an $i \in \{c+1, \ldots, N\}$ with $\tilde{q}_i(\theta) = 1$. By feasibility, $\tilde{t}_i(\theta) \geq \theta_i$. But then, $\theta_{c+1}(c+1) \leq \mathcal{B}$, a contradiction.

□

Proof of Lemma 11. As a proof of tightness, suppose $\mathcal{B} = 100$, and there are two sellers with $\theta_1 = 49$, $\theta_2 = 51$. The first-best allocation is $\{1, 2\}$, while

the auction selects $\{1\}$. Thus, the first-best allocation contains exactly twice as many elements.
In order to proof the 2-approximation property, suppose, w.l.o.g., the vector of types is ordered such that $\theta_1 < \theta_2 < \cdots < \theta_N$. Furthermore, suppose the allocation selected by the auction is $\{1, 2, \ldots, c\}$. We show that the first-best allocation cannot contain $2c+1$ elements (or more). Since $\mathcal{A} = \{1, 2, \ldots, c\}$, we have $\theta_{c+1}(c+1) > \mathcal{B}$, otherwise, by (65), seller $c+1$ would also be in $\mathcal{A}$. Suppose, per absurdum, that $\{1, 2, \ldots, c, \ldots, 2c+1\}$ is a subset of the first-best allocation, i.e., the first-best allocation has at least $2c+1$ elements. Then $\sum_{i=1}^{2c+1} \theta_i \leq \mathcal{B}$ which implies $\sum_{i=c+1}^{2c+1} \theta_i \leq \mathcal{B}$. Since the type vector is ordered, it follows that $\theta_{c+1}(c+1) \leq \mathcal{B}$ which implies that seller $c+1$ is in the allocation selected by the auction, a contradiction. $\square$

References

Aggarwal, G. and J. D. Hartline (2006): "Knapsack Auctions," *Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithms (SODA06)*, 1083–1092.

Börgers, T. and P. Norman (2009): "A note on budget balance under interim participation constraints: the case of independent types," *Economic Theory*, 39, 477–489, 10.1007/s00199-008-0347-7.

Chung, K.-S. and J. C. Ely (2002): "Mechanism Design with Budget Constraint," *Working Paper.*

Dizdar, D., A. Gershkov, and B. Moldovanu (2011): "Revenue Maximization in the Dynamic Knapsack Problem," *Theoretical Economics (forthcoming).*

Ensthaler, L. and T. Giebe (2009): "Subsidies, Knapsack Auctions and Dantzig's Greedy Heuristic," *SFB/TR 15 Discussion Paper No. 254.*

Giebe, T., T. Grebe, and E. Wolfstetter (2006): "How to allocate R&D (and other) subsidies: An experimentally tested policy recommendation," *Research Policy*, 35, 1261–1272.

Korte, B. and J. Vygen (2005): *Combinatorial Optimization: Theory and Algorithms*, 3th edition, Springer.

Maskin, E. (2010): "How to reduce greenhouse gas emissions - an application of auction theory," 2002 Nancy L. Schwartz Memorial Lecture, Northwestern University, private communication.

Myerson, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

Papadimitriou, C. and Y. Singer (2010): "Budget Feasible Mechanisms," *Working Paper (University of California at Berkeley).*