The Law of Attraction*

Bilateral Search and Horizontal Heterogeneity

Dirk Hofmann†

Humboldt-University Berlin

Salmai Qari‡

Max Planck Institute for Tax Law and Public Finance

July 8, 2011

Abstract
We study a matching model with heterogeneous agents, nontransferable utility and search frictions. Agents differ along a horizontal dimension (e.g. taste) and a vertical dimension (e.g. income). Agents' preferences coincide only with respect to the vertical dimension. This approach yields a tractable model to study the effects of individual preferences in labor markets (e.g. regional preferences) or marriage markets. Contrary to a model involving only a vertical dimension (e.g, Burdett and Coles 1997), agents continuously adjust their reservation utility strategies to changing search frictions. The model is easily generalizable in the utility specification, the distribution of taste-related payoffs and the number of vertical types.

Keywords: Matching, Horizontal Differentiation , Marriage Markets
JEL Classification Numbers: D13, D61, J12

*The authors are grateful to Helmut Bester, Tom Cusack, Kai Konrad and Roland Strausz for advice and suggestions. We are also grateful to audiences at the Free University Berlin, the Humboldt-University Berlin, the Social Science Research Center Berlin (WZB), the 2010 Econometric Society World Congress (Shanghai) and the Max Planck Research Center for Tax Law and Public Finance Munich for helpful comments.

†Humboldt-University Berlin, Institute for Microeconomic Theory, Spandauer Str. 1, 10178 Berlin, e-mail: *dirk.hofmann@staff.hu-berlin.de*

‡Max Planck Institute for Tax Law and Public Finance, Department of Public Economics, Marstallplatz 1, 80539 München, Germany, e-mail: *salmai.qari@tax.mpg.de*

1. Introduction

Finding a suitable match on the labor or marriage market is a complex task. Search frictions in the market force agents to trade off the chance of meeting a better partner in the future against the foregone utility due to staying single for a longer period of time. Quality is determined by a number of traits and agents usually agree on the ranking for a subset of the characteristics involved – the "vertical traits" – but disagree with respect to others – the "horizontal traits". In this paper, we study the different roles of vertical and horizontal heterogeneity among agents in matching models. The first parameter captures a discrete vertical dimension, for example income. All agents agree on the ranking along this vertical dimension. The second parameter represents a horizontal non-ordered characteristic. We refer to this trait as "taste" to highlight that agents do not necessarily agree on the ranking of agents along this dimension. Agents take the implications of both dimensions for their individual utility into account when deciding on their search strategy. The central aim of our paper is to employ a tractable model to analyze matching and sorting in this environment assuming that utility is nontransferable.

Our analysis is motivated by two distinct but closely related strands of literature. The first is the large empirical literature in economics and sociology documenting a positive association between traits of partners in existing unions. For example, the classic work by Becker (1973, 1974) finds a positive correlation between partners' education levels and partners' height. Kalmijn and van Tubergen (2006) provide empirical evidence for a correlation between ethnicity, while Kalmijn (2006) and Mare (1991) document assortative mating for religion and education. A more recent and important finding in this literature is that consumption complementarities with respect to horizontal traits, e.g, a shared interest in leisure activities, are becoming more and more important sources of the marital surplus (Lundberg and Pollak 2007, Lundberg 2011).

The second literature is the theoretical literature on two-sided search and matching. The seminal work by Becker (1973) analyzes in a static transferable-utility model how assortative mating arises in equilibrium. Starting from this basic model, a huge literature has emerged that discusses how different assumptions with respect to utility specifications, search frictions and heterogeneity change this outcome.[1] However, in contrast to the empirical literature the discussion of heterogeneity is centered on vertical traits.

[1] See Burdett and Coles (1999) for a survey of this literature.

If agents differ only with respect to a vertical dimension, all agents agree on the ranking of potential mates. Therefore, all individuals propose to the best available agent. Depending on the level of search frictions, this most desirable agent chooses the range of acceptable qualities. In this way, the boundary between the first and second class is derived. Similarly, the highest agent in the second class sets a range and hence creates the boundary between the third and second class (see, for example, Burdett and Coles 1997). The procedure continues until the equilibrium is derived for the whole range of the vertical trait. By contrast, since the agents in our model are additionally marked by a horizontal characteristic, there is no common most desirable agent. Although this complicates the analysis, our model and the corresponding equilibrium characterization remain tractable.

The interaction of horizontal and vertical preferences reveals some new insights in this strand of search and matching models. First, the model encompasses the prediction of assortative mating from the pure vertical model. If the level of search frictions is sufficiently low, the costs of rejecting offers are low. Therefore, agents, especially the vertically "gifted", are selective and this leads high type agents to reject low types. This in turn generates positive assortativeness.

Second, all agents adjust continuously their optimal reservation utility strategies in response to changing search frictions. This result seems particularly appealing compared to the "classical" model involving only the vertical dimension. In such a model agents might suffer from discrete reductions in expected utility if search frictions cross some (more or less) arbitrary level while increasing marginally.

Third, depending on the model parameters, low type agents might benefit from increasing search frictions. If the level of search frictions happens to be in such a range, a higher level of search frictions has two effects. First, the frequency of meeting potential partners is lower and this effect harms the agents. Second, the probability that a high type accepts a low type increases, and this effect outweighs the first effect.

Fourth, our analysis shows that both kind of matching equilibria, complete segregation of types and integrating equilibria may emerge. Finally, we show that our approach is general in the sense that it allows for (i) different utility specifications concerning the two traits, (ii) different type distributions along the horizontal dimension as well as (iii) for the number of vertical types. The analysis shows that extreme utility specifications (only the vertical dimension or respectively the horizontal dimension enters individual utility) are equivalent to a specification without vertical heterogeneity at all.

Our framework builds on the existing literature that deals with bilateral search and matching. As mentioned above, the seminal static model by Becker (1973, 1974) assumes that utility is transferable and that no search frictions are present. Recent analyses in the area of vertical heterogeneity and transferable utility are Bloch and Ryder (2000), Shimer and Smith (2000) and Atakan (2006). The papers that are closer to our analysis are Burdett and Coles (1997), Eeckhout (1999) and Smith (2006) who analyze vertical heterogeneity in a nontransferable utility model including search frictions.

Clark (2007) studies bilateral sorting with horizontal heterogeneity in a frictionless assignment model. We depart from Clark (2007) as in his model agents prefer similar agents, but the trait considered for matching is ordinal, e.g. height. The taste parameter in our model can be interpreted as agents' location on a circle. This standard approach in the Industrial Economics literature (Schmalensee 1978, Salop 1979) avoids end-point-effects. This assumption is also employed by Konrad and Lommerud (2010) who study the interplay of matching and redistributive taxation in a static model. In their model, agents are matched only for a single period in which they have to either accept a match or stay single forever.

We proceed as follows. In the following section we develop the marriage model. In the next section we solve for the equilibria in the model for one or two income-levels and discuss the comparative statics of the approach. Section 4 provides an example for specific income and taste distance functions. These permit an explicit model solution. Section 5 discusses three extensions of the model. First, we introduce weights for the two parts of utility into the model. This allows us to analyze the full range between two polar cases: in the first polar case agents marry for love and in the second they marry for money. Second, we introduce a reformulation that permits one to choose almost any continuous probability distribution for the utility along the horizontal dimension. Third, we show that our framework can be easily modified to account for more than two income levels. Finally, we discuss the implication of our analysis with respect to assortative matching. Section 6 concludes. All proofs are relegated to the appendix.

2. The Model

There is a unit mass of individuals. Each individual i is characterized by two parameters $(y_i, t_i) \in Y \times T$. The first parameter captures vertical ex-ante heterogeneity of agents. For concreteness, we refer to this trait as income y_i. We assume that there are two income levels such that $y_i \in Y = \{y_L, y_H\}$ with $y_H \geq y_L \geq 0$. The second parameter t_i captures horizontal heterogeneity and for concreteness we refer to it as taste t_i. We assume a continuum of taste parameters such that $t_i \in T = [0, 2]$.We follow the common approach from the Industrial Economics literature introduced by Schmalensee (1978) and Salop (1979), i.e. individuals are located around a circle of circumference 2.[2] This approach allows to avoid end point effects.[3] We assume that individuals are uniformly distributed on $Y \times T$; both income levels are equally likely and the location on the circle for each agent is drawn independently of her income.

Preferences of all agents are identical regarding the relative locations of potential partners to their own position. All agents prefer to be matched with partners of higher income. With respect to the horizontal dimension agents prefer to be matched with individuals of similar taste. If two agents i and j decide to marry each other, both enjoy a utility gain depending on a measure of the distance between their respective locations on the circle. Formally the distance in taste x between two agents i and j is given by

$$x := \begin{cases} \min\{t_i - t_j, 2 + t_j - t_i\}, & \text{if } t_i \geq t_j \\ \min\{t_j - t_i, 2 + t_i - t_j\}, & \text{if } t_i < t_j \end{cases}.$$

Hence, the taste difference of two randomly drawn agents x is the realization of a random variable X. Clearly, as both agents' taste parameters are drawn independently and are ex-ante uniformly distributed, the random variable X is uniformly distributed on $[0, 1]$.

Once a match is formed, individuals enjoy a utility flow depending on both partners' incomes and the taste difference x for all subsequent periods.[4] The utility function is additively separable. One part of an agent's utility is the gain induced by household income. A second source of utility is the intra-marital gain through the difference, or rather the similarity in taste. By this choice the two dimensions of utility generation become substitutes for the agents. If agent i marries agent j her per-period-utility equals

$$U(y_i, t_i, y_j, t_j) = f(y_i, y_j) + g(x),$$

where x is determined by the definition given above. Function f, which captures

[2]In contrast to Schmalensee (1978) and Salop (1979) we re-scale the circumference of the individual's location to 2. Hence, we have a maximum distance of 1 between two individuals on the circle.

[3]This differs from the approach by Clark (2007) who analyzes a model in which agents prefer agents of similar height. In such a model agents with "extreme" (small or large) height behave differently compared to agents with more common realizations.

[4]For simplification we assume that there is no divorce. This is a crucial assumption because agents stop searching in our model once they are married.

the part of agent i's utility induced by the vertical trait income, is assumed to be increasing in both arguments. Function g, which captures the (intra-marital) utility gain through the horizontal trait, is assumed to be differentiable in $(0, 1)$. Furthermore, we assume $g' < 0$, as agents prefer partners who are located closer to themselves. We normalize $g(1) = 0$.[5] As long as agent i stays single, her per-period-utility depends only on her income y_i and is given by

$$U(y_i, t_i) = f(y_i, y_i).^6$$

This definition of a single agent's utility reflects the idea that the marital surplus is generated by consumption complementarities with respect to the horizontal dimension (see, for example, Lundberg 2011). Furthermore, agent i's utility derived from income within marriage is only affected if she marries a partner with a different income level. For ease of notation and reading the utility of a single agent is sometimes written as $f(y_i, -) := f(y_i, y_i)$. We assume that marital surplus is not transferable.

The matching institution is characterized by search frictions, i.e. individuals face difficulties finding a spouse, a job or in general any type of matching partner. As commonly used in the matching literature with search frictions (e.g. Burdett and Coles 1997), α denotes the arrival rate of potential partners where α is the parameter of a Poisson process. The type of a potential partner j an individual i meets is assumed to be independent of individual i's type. If two singles meet, they first observe each others' traits and then decide either to propose to the other or to continue searching. If both singles propose to each other, a match is formed and the two singles leave the market forever, otherwise the agents continue searching. Two individuals forming a match are replaced by (unmarried) clones. Hence, the distribution of single agents is stationary across time. Future per-period utility flows are discounted at rate r.[7] Both the discount rate and the arrival rate of singles determine the extent of the market, which we denote by $\theta := \alpha/r$.

A strategy of a (single) individual in this framework is a subset of all potential partners for any point in time. Such a subset can be interpreted in the following way: The subset contains all partners this particular individual would like to marry and hence she would propose to at a certain point in time.[8] Such a strategy of a

[5] This normalization implies that there are no negative utilities induced by the maximum taste difference. This assumption does not affect the results qualitatively.

[6] Note that individual utility U is defined as a function of 4 arguments for married agents and as a function of only 2 arguments when for single agents.

[7] The life-time-utility of an individual marrying someone with per-period-utility k at period 0 when using discount factor r is $\int_0^\infty k \cdot e^{-rt} dt = \frac{k}{r}$.

[8] Note that agents *end* their game when marrying another agent as there is no divorce and

particular agent depends on her belief about the distribution of singles who propose to her. In turn, her strategy determines the offer distribution which other singles face. In equilibrium, beliefs and offer strategies must be compatible with each other. For example, consider for a moment two types of agents, females and males. Females have beliefs about the offers they receive from males and calculate a list of acceptable males based on these beliefs. This determines the offer distribution of females proposing to male agents. In equilibrium, females must have correct beliefs about the offers they receive from males. Likewise, male singles must have correct beliefs about the offers received by females.

In order to keep the model as tractable as possible, we focus entirely on a stationary environment. To be more precise, we assume that individuals employ time-stationary reservation utility strategies. We already noted that the distribution of singles is stationary over time due to the cloning assumption. Since all other agents as well employ time-stationary reservation utility strategies, the search problem of any particular agent is in fact stationary. It is therefore optimal for an individual agent to employ a stationary reservation utility strategy (Adachi 2003).

3. Optimal Search Policies

In the following, we first establish some intermediate results assuming that there is no vertical heterogeneity. This step-wise presentation of the model allows to develop an intuitive understanding of the effects due to horizontal heterogeneity. We will then analyze to what extent these intermediate results generalize to the case of two distinct income classes. Finally, we discuss comparative statics of the model.

3.1. No vertical heterogeneity

In this section we analyze agents' strategies and the resulting equilibria assuming that there is no vertical heterogeneity. Hence, all agents in the marriage market have the same income $y_H = y_L \equiv y$. As introduced above, the taste parameter t is distributed uniformly on $T = [0, 2]$. The taste parameter can be interpreted as the location of the respective agent on a circle with circumference 2.

Since the income-specific part of agents' utility is unaffected by marriage, single agents effectively condition their proposals to other singles only on taste. Furthermore, when choosing their optimal strategies, agents take the search frictions of the matching

consequently there are no strategic considerations for married agents.

institution into account. When agents have an offer at hand, they balance the possibility of receiving a better offer in future periods against the foregone utility due to staying single and continue searching.

The optimal strategy of an individual i is determined by using a dynamic programming approach. Agents maximize their expected lifetime utility V_i by choosing optimally the partners they would propose to.[9] For the next short time period Δ, lifetime utility V_i consists of three parts. First, for the duration of Δ the agent stays single and obtains $\Delta f(y, -)$. For the second term, let α_i denote the arrival rate of singles who are proposing to agent i. Moreover, let $H_{-i}(x)$ denote the probability that the difference in taste of a proposing agent is equal to x or less. Then, over this time interval of Δ she receives at least one proposal with probability $\Delta\alpha_i$. In this case the individual decides whether to accept or reject the proposal. Obviously, she will only accept proposals which yield a higher lifetime utility than V_i. Formally, utility resulting from a union with an agent of taste x exceeds V_i; $(f(y, y) + g(x))/r > V_i$. The third and last term of V_i captures the unlucky event that the agent did not meet anyone proposing to her in the time span of Δ and is stuck with her lifetime utility V_i.

Collecting terms and discounting yields

$$V_i = \frac{1}{1 + \Delta r} \left[\Delta f(y, -) + \Delta\alpha_i E_{-i} \left(\max \left\{ V_i, \frac{f(y, y) + g(x)}{r} \right\} \right) + (1 - \Delta\alpha_i)V_i \right] + o(\Delta) \tag{1}$$

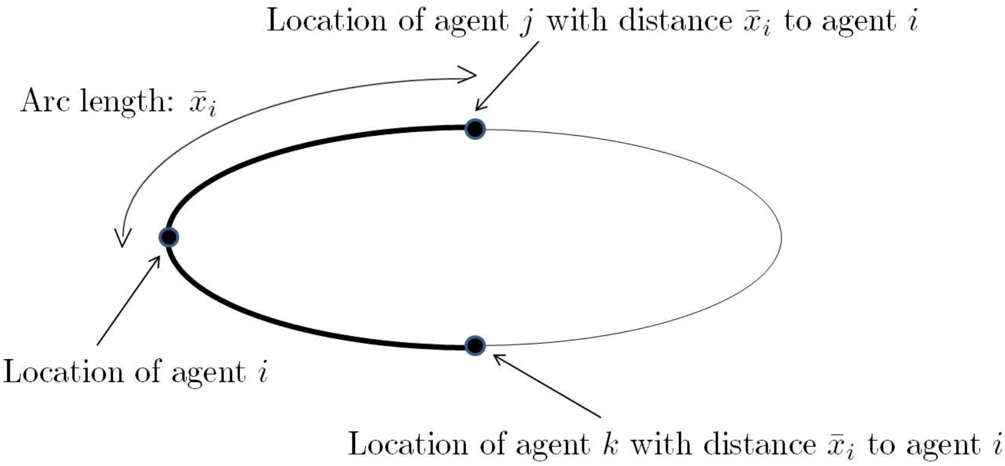
where agent i applies the expectation operator on the distribution of taste differences associated with the set of single agents proposing to her. Note that we follow the convention to subscript the expectations operator according to the respective random variable. Since agent i applies the expectation operator on taste difference $x \sim H_{-i}(x)$, it is denoted as E_{-i}. The o-function captures the small probability events of meeting more than one individual in time span Δ. Rearranging equation (1) and letting $\Delta \to 0$ yields

$$rV_i = f(y, -) + \frac{\alpha_i}{r} E_{-i} \left(\max \left\{ 0, f(y, y) + g(x) - rV_i \right\} \right). \tag{2}$$

As agents apply time-stationary reservation utility strategies and $g(\cdot)$ is decreasing in the taste difference x, equation (2) reveals that this corresponds to a cut-off strategy with respect to taste. Each individual i sets a certain minimal quality standard $\bar{x}_i$

[9] Burdett and Coles (1999) provide an introduction to the dynamic programming approach and a survey of the literature on search and matching.

Figure 1: Geometry of the One-Circle-Model



on her future partner and accepts proposals if and only if $x \leq \bar{x}_i$. We will refer to such strategies as reservation quality strategies. By this observation we restrict the further analysis to strategies of this type.

Rearranging equation (2) yields the following description of the agents' strategies:

Lemma 1 *Reservation utility strategies*

Given the expected distribution of offers by other agents $H_{-i}(x)$, agent i proposes to all singles with $x \leq \bar{x}_i$ where either

1. $rV_i = f(y, y) + g(\bar{x}_i)$ *and*

$$rV_i = f(y, -) + \frac{\alpha_i}{r} \int_0^{\bar{x}_i} \left(g\left(x\right) - g\left(\bar{x}_i\right)\right) dH_{-i}(x) \tag{3}$$

or

2. $rV_i < f(y, y) + g(\bar{x}_i)$, $H_{-i}(\bar{x}_i) = 1$ *and*

$$rV_i = f(y, -) + \frac{\alpha_i}{r} E_{-i}\left(f(y, y) + g(x) - rV_i\right). \tag{4}$$

Lemma 1 implicitly characterizes agent i's optimally chosen subset of acceptable agents, given her beliefs about the distribution of offers by other singles and about the search frictions in the market. In the first case agent i is decisive in the sense that she selects only a subset of all available offers. In the second case she is restricted by the offers made to her and is willing to accept all of them.

Figure 1 describes the type space and agent i's strategy. As there is only one income level, all agents are located on one circle. Agent i is located on the very left. The thick line in the figure represents agent i's cut-off-strategy. The partners she is willing to accept are symmetrically located around her own position. The offers agent i receives are not depicted in the figure. If agent i is decisive, it follows from Lemma 1 that she is indifferent between marrying and staying single when meeting the proposing agents j or k. If she is not decisive, agents j and i will not propose to agent i. In that case $H_{-i}(\bar{x}_i) = 1$ and agent i accepts all agents who propose to her.

Equilibrium

Lemma 1 states that the set of acceptable partners can be characterized as a uniform distribution with strictly positive support. That is, agent i accepts all individuals with $x \in [0, \bar{x}_i]$. Note that this formulation is silent about the offer distribution $H_{-i}(x)$. However, we will focus on symmetric equilibria in the sense that the offer distributions and acceptance sets are identical across agents. Hence, we rule out that the density h_{-i} is zero for some parameter range, while agent i's acceptance region is strictly positive on the full support. In the following μ_i denotes the fraction of singles who will propose to agent i on contact. The focus on symmetric equilibria is motivated by the fact that all agents are ex-ante symmetric in the sense that they have the same income level y and that they are distributed uniformly on the circle. Hence, there is ceteris paribus no agent who prefers moving to a different location.

The focus on symmetric equilibria implies that any agent i is decisive in the sense of the first part of Lemma 1, which allows us to rewrite equation (2) as

$$rV_i = f(y, -) + \frac{\alpha\mu_{-i}}{r} \int_0^{\bar{x}_i} \left(g(x) - g(\bar{x}_i)\right) h_{-i}(x) dx. \tag{5}$$

The offer distribution is strictly positive on the full support and individuals are located uniformly on the circle. Therefore, the offer distribution is a uniform distribution with associated density $h_{-i}(x) = 1/\mu_{-i}$. Inserting this density into equation (5) yields[10]

$$rV_i = f(y, -) + \theta \left[\int_0^{\bar{x}_i} g(x) dx - \bar{x} g(\bar{x}) \right]. \tag{6}$$

[10]Note that $\int_0^{\bar{x}} g(x) - g(\bar{x}) dx = \int_0^{\bar{x}} g(x) dx - \bar{x} g(\bar{x})$.

Furthermore, the first part of Lemma 1 yields

$$rV_i = f(y, y) + g(\bar{x}_i). \tag{7}$$

Equilibrium requires that equation (6) and equation (7) are satisfied for any agent i. By construction of the model, V_i is the maximum value of lifetime utility for each agent i. Hence, all agents face the same decision problem and will apply the same cut-off-value $\bar{x} = \bar{x}_i$ and receive the same lifetime utility $V_i = V$. Rewriting equations (6) and (7) yields the solution for the equilibrium that we summarize in the following proposition.

Proposition 1 *Equilibrium without vertical heterogeneity*
In equilibrium, all agents marry the first single who yields at least the reservation utility rV, where rV is characterized by:

$$\begin{array}{ll} (I) & rV = f(y, y) + g(\bar{x}) \\ (II) & rV = f(y, -) + \theta \left[\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x}) \right] \end{array}$$

Proposition 1 fully characterizes the two unknown parameters ($\bar{x}$ and V) for a given extent of the market θ.[11] For ease of exposition, we will mainly discuss the features of the model in terms of the quality $\bar{x}$. The following corollary summarizes the main properties.

Corollary 1 *Properties of the reservation quality*

[11]We can easily rewrite the model to a setup with two sets of agents, say females and males. In this case let $i \in \{w, m\}$ denote women and men respectively. The reservation utility for females and males is given by

$$rV_w = f(y, -) + \frac{\alpha \mu_m}{r} \int_0^{\bar{x}_w} (g(x) - g(\bar{x}_w))\, h_m(x)dx \tag{8}$$

and

$$rV_m = f(y, -) + \frac{\alpha \mu_w}{r} \int_0^{\bar{x}_m} (g(x) - g(\bar{x}_m))\, h_w(x)dx \tag{9}$$

respectively. In equilibrium the arrival rate of offers μ_m and the conditional taste distribution $H_m(x)$ must be consistent with male agents' reservation strategy defined by equation (9). Likewise, $H_w(x)$ and μ_w must be consistent with females' strategy given by equation (8).

1. The reservation quality $\bar{x}$ is characterized by

$$g(\bar{x}) = \theta \left[\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x}) \right]. \tag{10}$$

2. The reservation quality $\bar{x}$ is decreasing in θ and we have

$$\lim_{\theta \to 0} \bar{x} = 1 \text{ and } \lim_{\theta \to \infty} \bar{x} = 0.$$

As there is no intra-marital income redistribution, the reservation quality is unaffected by the income level y. In general, all agents prefer to be matched with singles of similar taste, as such matches yield the largest utility gain. An extent θ close to zero implies that there are maximum search frictions in the market.[12] In this case all singles set $\bar{x}$ to the maximum and accept the first single they encounter. On the other hand, a large extent θ implies that singles can easily meet each other or that they are very patient. For $\theta \to \infty$ there are no search frictions, and every agent will "wait" (for a time span of zero) for her perfect match. This corresponds to the perfect assortative mating results in previous models without search frictions, e.g. Becker (1973). However, this "notion of assortative mating" differs from the usual definition of assortative mating, as the taste parameter is not an ordinal measure. Assortative mating can be defined with respect to correlation between spouses' location on the circle. If search frictions are absent, agents complete mating only with individuals of the same taste parameter. Hence, there is perfect correlation between partners' location. For maximum search fractions there is no correlation between spouses' taste parameters. For intermediate levels of search frictions, the correlation with respect to taste is decreasing in the level of search frictions.

3.2. Introducing vertical heterogeneity ("Two circles")

In this section we introduce vertical heterogeneity into the model. Therefore we drop the assumption of a single income level and consider two income levels $y_H > y_L$. These two income levels are equally likely in the population. As before, the taste parameter t is distributed uniformly on $T = [0, 2]$ and is drawn independently of an agent's income. By this specification we have two different types of agents: High type agents with high income y_H and low type agents with low income y_L. Whenever it is necessary to distinguish between one particular agent and types of agents, we

[12] This is due to the normalization $g(1) = 0$. Otherwise agents would accept only those partners who are associated with a non-negative marital surplus.

denote individuals by small letters and types by capital letters.

Contrary to the case of a single income level, agent i takes into account possible intra-marital income redistribution as $f(y_i, y_H) > f(y_i, y_L)$. As before, we solve for the symmetric equilibrium of the extended model. Once again agents balance waiting costs and the foregone utility of marrying a mediocre partner. When evaluating possible partners, both characteristics are to some extent substitutes: Agents may be willing to accept lower income for a better match in taste and vice versa.

In principle, we derive lifetime utility of a particular agent analogously to equation (1). However, each agent now faces two different offer distributions: Offers received from high type agents and offers from low type agents. For high type agents who are willing to marry agent i who has income y_i, let $H^H_{-i}(x_H|y_i)$ denote the cumulative probability that the difference in taste is equal to x_H or less. Analogously, we define $H^L_{-i}(x_L|y_i)$ and x_L for low types.[13] Let α^H_i denote the probability that agent i contacts a high type single who is willing to marry her. Likewise, let α^L_i denote the probability that she contacts a low type single who is willing to marry her. As in the case without vertical heterogeneity, agents employ reservation utility strategies. Using the definitions so far, we can extend equation (1) to account for the two types. Hence, expected discounted lifetime utility of agent i is given by

$$\begin{aligned} V_i = \frac{1}{1+\Delta r} \Bigg[& \Delta f(y_i, -) \\ & + \Delta(\alpha^H_i + \alpha^L_i) \left(\sum_K \frac{\alpha^K_i}{\alpha^H_i + \alpha^L_i} E^K_{-i} \left(\max \left\{ V_i, \frac{f(y_i, y_K) + g(x_K)}{r} \right\} \right) \right) \\ & + (1 - \Delta(\alpha^H_i + \alpha^L_i)) V_i \Bigg] + o(\Delta) \end{aligned} \tag{11}$$

where $K \in \{H, L\}$.

Rearranging equation (11) and letting $\Delta \to 0$ yields

$$rV_i = f(y_i, -) + \sum_K \frac{\alpha^K_i}{r} E^K_{-i} \left(\max \{0, f(y_i, y_K) + g(x_K) - rV_i\} \right). \tag{12}$$

Equation (12) implies that agent i optimally chooses cut-off-values with respect to the difference in taste. In particular, agent i now calculates two reservation taste levels $\bar{x}_i(\cdot)$ depending on the income type of proposing agents. Analogously to Lemma 1, we characterize the reservation-utility strategies as follows:

[13] To stress that the offer distribution of agent i is fundamentally determined by her income type y_i, we write $H^K_{-i}(\cdot|y_i)$ instead of the (rather correct) $H^K_{-i}(\cdot)$ for $K \in \{H, L\}$.

Lemma 2 *Reservation utility strategies for two income levels*

Given the expected distribution of offers by other agents $H_{-i}^{K}(x_K|y_i)$ *where* $K \in \{H, L\}$*, agent i uses a reservation taste strategy and accepts all agents with* $x \leq \bar{x}_i(y_K)$ *where either*

1. $rV_i = f(y_i, y_K) + g(\bar{x}_i(y_K))$ *and*

$$rV_i = f(y_i, -) + \sum_K \frac{\alpha_i^K}{r} \int_0^{\bar{x}_i(y_K)} \left(g\left(x_K\right) - g(\bar{x}_i\left(y_K\right))\right) dH_{-i}^{K}\left(x_K|y_i\right) \tag{13}$$

or

2. $rV_i < f(y_i, y_K) + g(\bar{x}_i(y_K))$, $H_{-i}^{K}(\bar{x}_i(y_K)) = 1$ *and*

$$rV_i = f(y_i, -) + \sum_K \frac{\alpha_i^K}{r} E_{-i}^{K}\left(f(y_i, y_K) + g(x_K) - rV_i\right). \tag{14}$$

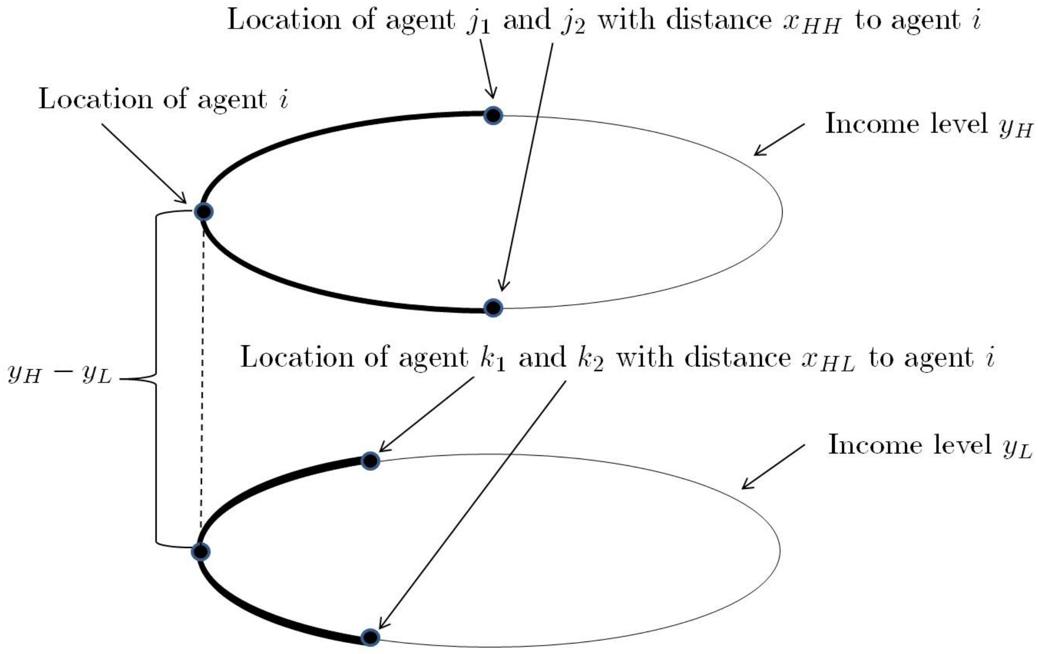
Equilibrium

As in the previous section we focus on symmetric equilibria. All agents with the same income are ex-ante identical due to our assumptions on the distributions of individuals' traits. But compared to the case without vertical heterogeneity, there are now two different types of agents, high types and low types. By imposing symmetry on the equilibrium outcome we require that agents of the same income type apply the same strategies.

While the solution for the model without vertical heterogeneity is characterized by two equations for the two endogenous variables V and $\bar{x}$, there are now in principle 6 endogenous variables. These are the expected discounted lifetime utilities for both high and low type agents (V_H and V_L) and the reservation qualities for type K_1 agents proposing to type K_2 agents. The reservation qualities are denoted by $x_{K_1K_2}$, where $K_1, K_2 \in \{H, L\}$. For example, the *least acceptable agent* for high types proposing to low types is denoted by x_{HL}.

Figure 2 shows the basic geometry of the model. There are two income levels $y_H > y_L$ and hence agents are located on two circles. Agents are distributed uniformly on the two circles, while the fraction of high and low type agents is equal. For example, agent i is assumed to have high income y_H and is located to the very left. For any given offer distributions from high and respectively low types, her best response is a cut-off-strategy. The figure depicts the two reservation qualities x_{HH} and x_{HL}. As partners with high income are more attractive compared to their low income

Figure 2: Geometry of the Two-Circles-Model



counterparts (with identical taste), agent i is less selective among rich types, i.e. we have $x_{HH} \geq x_{HL}$.

As a first step in solving the equilibrium, the set of initially six relevant unknown variables is reduced. A simple observation shows that in equilibrium high types will have a higher reservation utility than low types, that is we have $V_H > V_L$. Suppose to the contrary they would have a lower reservation utility than their low type counterparts. Then, a high type could deviate to the strategy of a low type. As she is even more attractive than the low type due to her income, she receives at least as many offers as the low type agents. Hence, her utility stemming from possible future marriages is at least as high as a low type's utility from marriage events (as all agents use reservation utility strategies). In addition, her utility associated with staying single is higher. This shows that in equilibrium we must have $V_H > V_L$ as $y_H > y_L$. Consequently, high types are more selective with respect to low types than low types are with respect to high types. Thus, we have $x_{HL} \leq x_{LH}$. This implies that x_{LH} does not matter for equilibrium considerations, as low type agents are not decisive for mixed marriages.

Depending on the level of search frictions two distinct types of equilibria emerge; equilibria with and respectively without income segregation. To understand the mechanism that drives this result, assume that the extent of the market is very large ($\theta \to \infty$). Intuitively, in this case a high type agent would even reject a poor type who has the same taste as her. That is, an agent with characteristics (y_H, t_i) rejects the agent with characteristics (y_L, t_i) because of intra-marital income redistribution. Hence, high type agents generally reject low type agents and an income segregation equilibrium emerges. Only below a certain threshold $\tilde{\theta}$, high type agents will consider marrying low type agents. For all extents θ above this threshold the equilibrium solution exhibits $x_{HH} > 0$ and $x_{HL} = 0$. We will discuss the characteristics of $\tilde{\theta}$ after establishing the complete solution for the two types of equilibria.

We start the equilibrium analysis with the case $\theta < \tilde{\theta}$, where mixed marriages occur in equilibrium. First, we look at the equilibrium strategy of a high type agent i. As mentioned above, all agents apply reservation-utility strategies and we focus on symmetric equilibria. Hence, both acceptance and offer distributions are characterized by a non-degenerate density function with strictly positive support. Furthermore, as $x_{HL} \leq x_{LH}$, we know that a high type agent i is decisive. Let $H^K_{-i}(x_K|y_H)$ denote the offer distributions of agents of type $K \in \{H, L\}$, where μ^K_{-i} denotes the fraction of type K-singles who are willing to marry agent i on contact. As both high and low types are equally likely, this implies that agent i faces an arrival rate of $0.5\alpha\mu^H_{-i}$ from high types and respectively $0.5\alpha\mu^L_{-i}$ from low types. Exploiting the decisiveness of agent i, we apply the first part of Lemma 2 which yields

$$rV_i = f(y_i, -) + \sum_K \frac{0.5\alpha\mu^K_{-i}}{r} \int\limits_0^{\bar{x}_i(y_K)} \left(g\left(x_K\right) - g(\bar{x}_i\left(y_K\right))\right) dH^K_{-i}\left(x_K|y_i\right). \quad (15)$$

Similar to the derivation involving only a single income level, the relevant densities are $h^K_{-i}(x_K|y_i) = 1/\mu^K_{-i}$. Since agent i is a high type, we employ the type-based notation for high types and rewrite equation (15) as follows

$$rV_H = f(y_H, -) + \frac{1}{2}\theta \left[\int\limits_0^{x_{HH}} g(x)dx + \int\limits_0^{x_{HL}} g(x)dx - g(x_{HL})x_{HL} - g(x_{HL})x_{HL} \right]. \quad (16)$$

Applying once more agent i's decisiveness, the first part of Lemma 2 yields:

$$rV_H = \quad f(y_H, y_H) + g(x_{HH}) \quad (17)$$

$$rV_H = \quad f(y_H, y_L) + g(x_{HL}) \quad (18)$$

The three equations (16), (17) and (18) provide a full characterization of the first three variables of the model V_H, x_{HH} and x_{HL}. Equations (17) and (18) provide a further intuitive insight as

$$g(x_{HL}) - g(x_{HH}) = f(y_H, y_H) - f(y_H, y_L).$$

This result shows that under any specification of search frictions a high type agent "requires" from low type agents a better match with respect to taste. More specifically, this compensation is a constant defined through the utility specification. The difference between the reservation qualities measured in g equals the income loss due to redistribution.

Now, we turn to the solution for a low type agent j. Recall that $x_{LH} \geq x_{HL}$, as high types are decisive for mixed marriages. Hence, employing once again the symmetry argument, agent j is only decisive within other poor agents. Consequently, we employ both parts of Lemma 2 to calculate the reservation utility for agent j. For low types proposing to agent j, part (i) of Lemma 2 applies, while part (ii) is suitable for high types proposing to agent j. Collecting terms yields

$$\begin{aligned} rV_j = f(y_j, -) &+ \frac{\alpha_j^H}{r} E_{-j}^H \left(f(y_j, y_H) + g(\bar{x}_j(y_H)) - rV_j \right) \\ &+ \frac{\alpha_j^L}{r} \int_0^{\bar{x}_j(y_L)} \left(g\left(x_L\right) - g(\bar{x}_j\left(y_L\right)) \right) dH_{-j}^L \left(x_L | y_j\right). \end{aligned} \tag{19}$$

The fact that high types are decisive for mixed marriages permits a step-wise equilibrium derivation. The reservation utility strategies for any high type agent i is already determined. This includes the threshold quality x_{HL}. Hence, the fraction of high type agents proposing to low type agents μ_{-j}^H is given and equal to x_{HL}. This simplifies the arrival rate of offers from high types to $\alpha_j^H = 0.5\alpha x_{HL}$. Furthermore, the density of $H_{-j}^L\left(x_L|y_j\right)$ is $h_{-j}^L = 1/\mu_{-j}^L$.

For a low type agent j, the reservation utility with respect to other low types is by Lemma 2

$$rV_L = \quad f(y_L, y_L) + g(x_{LL}). \tag{20}$$

Using these simplifications and exploiting the type-based notation, equation (19) simplifies to

$$rV_L = f(y_L, -) + \frac{1}{2}\frac{\alpha}{r} x_{HL} \left[f(y_L, y_H) + E\left(g(x) \mid x \leq x_{HL}\right) - f(y_L, y_L) - g(x_{LL})\right] + \frac{1}{2}\frac{\alpha}{r} \left[\int_0^{x_{LL}} g(x)dx - x_{LL} g(x_{LL}) \right]. \tag{21}$$

This completes the equilibrium solution. The following proposition collects the relevant equations and summarizes the results for $\theta < \tilde{\theta}$.

Proposition 2 *Integration equilibrium*

The solution for the five unknown endogenous variables is characterized by the following system of equations:

$$(I)\; rV_H = f(y_H, y_H) + g(x_{HH})$$

$$(II)\; rV_H = f(y_H, y_L) + g(x_{HL})$$

$$(III)\; rV_H = f(y_H, y_H) + \frac{1}{2}\frac{\alpha}{r} \left[\int_0^{x_{HH}} g(x)dx + \int_0^{x_{HL}} g(x)dx - x_{HH} g(x_{HH}) - x_{HL} g(x_{HL}) \right]$$

$$(IV)\; rV_L = f(y_L, y_L) + g(x_{LL})$$

$$(V)\; rV_L = f(y_L, y_L) + \frac{1}{2}\frac{\alpha}{r} x_{HL} \left[f(y_L, y_H) + E\left(g(x) \mid x \leq x_{HL}\right) - f(y_L, y_L) - g(x_{LL})\right] + \frac{1}{2}\frac{\alpha}{r} \left[\int_0^{x_{LL}} g(x)dx - x_{LL} g(x_{LL}) \right]$$

Equation (III) of the solution captures the standard reservation equation for high type agents. It implies that the flow value of search rV_H is equal to the current payoff $f(y_H, -) = f(y_H, y_H)$ plus the expected surplus generated by the optimal search strategy. This optimal strategy is given by x_{HH} and x_{HL}. Equations (I) and (II) show another standard reservation strategy result: High agents must be indifferent between searching further and marrying the (optimally chosen) least acceptable partner among high types (in taste distance x_{HH}) and among low types (in taste distance x_{HL}). In addition the two equations capture the constraints for this strategy discussed above; that is high type agents require compensation along the horizontal dimension from low type agents.

Equation (V) describes the standard reservation equation for low type agents. Since low type agents are not decisive for mixed marriages, they treat x_{HL} essentially as given. Therefore, the first bracket of the right hand side can be interpreted as the current payoff including a stochastic component generated by possible mixed marriages. The second bracket equals the expected surplus from choosing the optimal reservation quality within their own types. Low types have only one decision variable and hence equation (IV) completes the solution. This equation describes the usual indifference condition and is comparable to equations (I) and (II).

As mentioned above, the fact that high types are decisive for mixed marriages permits a step-wise solution for the equilibrium. Low types take the high types' decisions in equilibrium as given and optimize actively only within their own types. This property has simplified the derivation of Proposition 2, and the extensions section of this paper shows how this concept generalizes to any (discrete) number of agents.

We now turn to the case $\theta > \tilde{\theta}$. In this case search frictions are low such that high types are selective and reject offers from low types. As a result, the two classes endogenously partition themselves into two classes and choose the threshold quality within their own class. Therefore, there are no mixed marriages and we can immediately apply the results from the case without vertical heterogeneity.

To solve for the symmetric equilibrium we use Proposition 1 for the two types of agents separately. Compared to the case without vertical heterogeneity in section 2, we have to adjust the arrival rate of offers. Since both types are equally likely and the assumptions with respect to random matching still hold, the adjusted arrival rate is now 0.5α. The following proposition summarizes the results for the segregation equilibrium:

Proposition 3 *Segregation equilibrium*

The solution for the three endogenous variables is characterized by the following system of equations:

$$\begin{aligned} g(\bar{x}) &= \frac{\theta}{2}\left[\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x})\right] \\ rV_H &= f(y_H, y_H) + g(\bar{x}) \\ rV_L &= f(y_L, y_L) + g(\bar{x}) \end{aligned}$$

The first equation characterizes the reservation quality $x_{LL} = x_{HH} = \bar{x}$. Both agents use the same reservation quality, because there are no mixed marriages and hence there is no intra-marital income redistribution. But clearly, the utilities of either type differ according to the different income levels. This can be seen in Proposition

3 as $f(y_H, y_H) > f(y_L, y_L)$.

Finally, we solve for the critical level of search frictions $\tilde{\theta}$ that separates the integration equilibria from the segregation equilibria. When the level of search frictions is equal to $\tilde{\theta}$, a high type agent is indifferent between accepting the best available low type agent and the optimally chosen high type agent. Hence, equations (I) and (II) from Proposition 2 yield

$$f(y_H, y_H) + g(x_{HH}) = f(y_H, y_L) + g(0). \quad (22)$$

As equation (22) shows, the threshold extent translates directly into a reservation quality among high types $\tilde{x}_{HH}$. Setting $x_{HL} = 0$ in equation (III) from Proposition 2 yields this threshold reservation quality $\tilde{x}_{HH}$. It is easy to show that this equation and the first equation from Proposition 3 coincide for $x_{HL} = 0$. Hence, the threshold extent is well-defined. This completes the solution.

Properties of the solution

In the following we discuss some properties with respect to the solution of the general model with vertical heterogeneity and provide some intuition for the results developed so far. The relevant exogenous parameter for equilibrium behavior is the extent θ. We therefore write particular endogenous variables of the model as functions of the extent θ and discuss properties of the equilibrium solution with respect to changes in θ.

We start the discussion with some continuity properties of the solution:

Proposition 4 *Continuity Properties*

The reservation qualities $x_{HH}(\theta), x_{HL}(\theta)$ and $x_{LL}(\theta)$ are continuous in $\theta \in \mathbb{R}_+$.

The continuity for $\theta \neq \tilde{\theta}$ is straightforward as the optimal cut-off strategies are adjusted smoothly by the agents. When search frictions increase (θ decreases) starting from $\tilde{\theta}$, high type agents start sending offers to low types. The possible utility gains with respect to the horizontal dimension are distributed in a continuous way on the two circles. Therefore, there is continuity for the reservation utility strategies of high type agents with respect to changes in θ.

From the perspective of low types, the first proposals from high types slightly below $\tilde{\theta}$ start with probability zero. This probability increases continuously if θ decreases further. Hence, the reservation utility of low types is not affected in a discontinuous manner. It follows that there is no discontinuity in their offer behavior within their own types.

Finally, for the reservation quality that high types require from low types we have already established that $x_{HL} = 0$ and hence continuity is obvious.

The continuity property distinguishes our model from the dynamic search models involving only vertical heterogeneity. For example, the approaches by Burdett and Coles (1997), Eeckhout (1999) and Smith (2006) share the common feature that agents partition themselves into a certain number of classes. In equilibrium, all agents within a particular class obtain identical (reservation) utilities. Search frictions mainly influence the size of these classes. Intuitively, the partitioning into distinct classes allows agents to pool with other agents within their class. The best available agent within this class is the agent whose vertical characteristic is equal to the upper bound of the interval characterizing the highest class. All agents below the best available agent benefit from pooling with this agent, because they face the same probability of receiving an offer from such an agent.

However, with respect to the example above, some agents might drop out of the highest class when search frictions increase. The agents who drop out and move to the second highest class are suddenly not decisive for marriages with class-1-agents. This change happens when search friction marginally increase starting at some threshold. Hence, the reservation utility of these agents sharply drops. This generates a discontinuity which seems to be rather questionable from an applied perspective. Our approach results in a continuous adjusting of reservation utility strategies. This is driven by the fact that agents are at least decisive for some marriages within their own vertical class. In effect, they adjust strategies continuously whenever search frictions decrease.[14] Moreover, agents with different vertical traits always have different lifetime utilities.

We proceed by analyzing the limiting behavior of the different reservation qualities:

Proposition 5 *Limiting Properties*

The limiting behavior of the reservation qualities is as follows:

a) $\lim_{\theta\to 0} x_{HH}(\theta) = 1, \lim_{\theta\to 0} x_{LL}(\theta) = 1$ *and* $\lim_{\theta\to 0} x_{HL}(\theta)$ *is given by* $g(x_{HL}) = f(y_H, y_H) - f(y_H, y_L)$.

b) $\lim_{\theta\to\infty} x_{HH}(\theta) = 0, \lim_{\theta\to\infty} x_{LL}(\theta) = 0$

[14] This feature of our model is *not* driven by reducing the number of (vertical) types to two. We will return to this point when we extend the model to the N-type case.

The first part of Proposition 5 covers the case of maximum search frictions. The normalization $g(x) \geq 0$ implies that agents (weakly) prefer to marry any partner of their own class rather than staying single. Since $x \in [0, 1]$, the limiting threshold quality for maximum search friction is equal to one.

As a high type faces redistributional losses when marrying a low type, she is only willing to marry low type agents who compensate her for this loss. This holds as well for maximum search frictions and explains the limiting behavior of $x_{HL}(\theta)$.

The second part of Proposition 5 is straightforward. In a market without search frictions complete segregation emerges. Every agent sets her required minimum quality to the maximum as she meets all agents instantaneously.

We now discuss the differentiability properties that we summarize as follows:

Proposition 6 *Differentiability Properties*

a) Reservation quality $x_{LL}(\theta)$ is differentiable for all $\theta \in \mathbb{R}_+ \setminus \tilde{\theta}$ and is not differentiable in $\tilde{\theta}$.

b) Reservation quality $x_{HH}(\theta)$ is differentiable for all $\theta \in \mathbb{R}_+$ and is decreasing in θ.

c) Reservation quality $x_{HL}(\theta)$ is differentiable for all $\theta \in \mathbb{R}_+ \setminus \tilde{\theta}$ and is non-increasing in θ.

Once again, the properties for $\theta \neq \tilde{\theta}$ are straightforward. Agents smoothly adjust their reservation utility strategies in response to a small change in search frictions.

By contrast, around $\tilde{\theta}$ only x_{HH} is differentiable. When search frictions increase (θ decreases) beyond $\tilde{\theta}$, high types start proposing to low type agents. Contrary to low types, high types are decisive. As marriages with low types start out as a probability zero event, high types do not need to adjust the rate of their proposals to other high types when search frictions increase. Hence, the solution for x_{HH} exhibits differentiability in $\tilde{\theta}$.

In general, reservation qualities x_{HL} and x_{LL} are not differentiable in $\tilde{\theta}$. For an extent slightly below $\tilde{\theta}$, low types receive offers from high types with a positive probability. Low type agents receive a larger utility gain from these offers than from the least acceptable partner within their own class. Since the low types are not decisive for these mixed marriages, one can interpret the possibility of marrying a high type as a new opportunity which happens by chance. Since low types correct their reservation quality for these lucky events, their reservation quality is not differentiable in $\tilde{\theta}$. An example which illustrates this is discussed in section 4. The example also shows that low types may even benefit from higher search frictions because of these lucky events.

3.3. Comparative Statics

As the agents do not agree on the ranking with respect to the horizontal dimension, the type of equilibrium may change if weights are introduced. To analyze this question, we now introduce weights that capture the idea that –for example– a population is more concerned about potential spouses' taste than spouses' income.

In particular, let $\beta \in (0, 1)$ denote the weight for the income-related payoff and let $1 - \beta$ be the weight on taste-related utility. Agent i's utility if married to agent j is then defined as

$$U(y_i, t_i, y_j, t_j) = \beta f(y_i, y_j) + (1 - \beta)g(x),$$

where x is the taste difference as before.[15] The limiting cases $\beta \to 0$ and $\beta \to 1$ describe agents who are "romantically minded" or "greedy for money" respectively.[16]

First note that these weights do not change the general analysis at all. As utility is ordinal, we can simply transform the utility function by dividing with weight β. This transformed utility function is

$$\tilde{U}(y_i, t_i, y_j, t_j) = f(y_i, y_j) + \tilde{g}(x) \text{ with } \tilde{g}(x) = \frac{(1 - \beta)}{\beta} g(x).$$

As $\tilde{g}$ is decreasing and $\tilde{g}(1) = 0$, the transformed utility $\tilde{U}$ meets all requirements for the utility specification, that are discussed in section 2. Hence, the analysis does not change and replacing g by $\tilde{g}$ in Proposition 1 yields the solution for the case without vertical heterogeneity. Likewise, Proposition 2 and Proposition 3 implicitly characterize the solution for the integration and segregation equilibria respectively.

In the following, we discuss the comparative statics for the reservation qualities associated with the limiting cases $\beta \to 0$ and $\beta \to 1$. By equations (I) and (II) of Proposition 2, the critical market extent $\tilde{\theta}$, that divides the integration equilibrium from the segregation equilibrium (where $x_{HL} = 0$), is implicitly defined by

$$\beta(f(y_H, y_H) - f(y_H, y_L)) = (1 - \beta)(g(0) - g(x_{HH})). \tag{23}$$

Note that the critical market extent is well defined by inserting the modified utility

[15] Hence, there is now only one additional normalization required; as before we assume $g(1) = 0$. Once again, this assumption is only for the sake of simplicity and could be replaced by constant.

[16] We stick to the assumption that all agents have the same utility function.

specification into equation (22). As before, equation (23) yields an expression for x_{HH} at the critical market extent $\tilde{\theta}$. Clearly, an increase of β requires a higher threshold x_{HH} at $\tilde{\theta}$ as g is decreasing in x. What are the implications for $\tilde{\theta}$? From the previous analysis we know that the high types' reservation qualities are increasing in θ, i.e. the lower the search frictions (increasing θ) the lower the taste difference for the least acceptable partners (e.g. $\bar{x}$ is decreasing). Hence, this shows that higher values of β are associated with smaller values of the critical market extent $\tilde{\theta}$.

Using this result, it is straightforward to show that the integration equilibrium does not exist for values of β beyond an upper bound $\tilde{\beta}$. Note that the right-hand-side of equation (23) captures the maximum possible gain along the horizontal dimension relative to the least acceptable high type partner. It is bounded by $(1-\beta)g(0)$ as $g(1) = 0$. Recall that the left-hand-side of this equation describes the income loss for the high type agent due to intra-marital redistribution when marrying a low type. If this loss exceeds the maximum possible gain with respect to the horizontal dimension, high type agents reject any low type agents. Since the maximum gain equals $g(0)$, the upper bound for β is characterized by the following inequality

$$\beta(f(y_H, y_H) - f(y_H, y_L)) > (1-\beta)g(0).$$

Rearranging shows that for

$$\beta > \tilde{\beta} := \frac{g(0)}{f(y_H, y_H) - f(y_H, y_L) + g(0)}$$

high types will reject low type agents for all values of θ.

It remains to discuss the optimal strategy of high and low types within their own class. Since there is no intra-marital redistribution for these marriages, the two types of agents apply the same cut-off-value with respect to taste. To summarize, for

$$\beta > \tilde{\beta}$$

low and high types optimize in separate spheres. The equilibrium is fully characterized by inserting $\tilde{g}(x)$ and $x_{HL} = 0$ into Proposition 3. Once again, since complete segregation emerges, the arrival rate of potential partners who have the same income equals $\alpha/2$. Finally, note that $0 < \tilde{\beta} < 1$.

Obviously, decreasing β is associated with larger values of $\tilde{\theta}$. In the limiting case, income becomes irrelevant for the agents' decision problem as in the case without vertical heterogeneity. Hence, the functions x_{HH}, x_{HL} and x_{LL} converge pointwisely to the same function which is the one-circle solution for an arrival rate of α.

In sum, we have established two results. First, assigning weights to the two parts of the utility function does not change the analysis. The solution is obtained by a simple reformulation of the variables. More interestingly, for both limiting cases the resulting model solution can be derived by exploiting the available results for the case without vertical heterogeneity. However, the critical threshold $\tilde{\theta}$ differs across the two limiting cases.

The reasoning for the two limiting cases is also quite different. Intuitively, for large values of β only high types are acceptable for a high type. Hence, low types are stuck with each other and this leads to complete segregation. Since the markets of low and high type agents are divided into two separate parts, the arrival rate of offers is equal to 0.5α. Hence, the reservation qualities for this case can be obtained by inserting this modified arrival rate into the one-circle-solution. By contrast, for low values of β income is not very important and in the limiting case ($\beta \to 0$) income does not affect agents' decisions. In this case both low and high type agents are basically identical. Hence, the result converges to the one-circle solution with the original arrival rate α.

4. An Example

We now illustrate the mechanics of the model by a particularly simple example that permits the derivation of an explicit solution. First, we assume that utility derived from the vertical dimension is equal to the average household income if agents are married. As before, single agents' utility equals their own income. Formally, we specify $f(\cdot,\cdot)$ as follows:

$$f(y_H, y_H) = f(y_H, -) = y_H \tag{24}$$

$$f(y_H, y_L) = f(y_L, y_H) = \frac{y_H + y_L}{2} \tag{25}$$

$$f(y_L, y_L) = f(y_L, -) = y_L \tag{26}$$

This assumption might be interpreted as full income pooling. Both spouses receive the same income share independently of their contributed shares to household income. With respect to the taste dimension, we assume that utility is decreasing in the simple linear distance. Formally, we have

$$g(x) = 1 - x.$$

Clearly, this specification is in line with the set of assumptions from the previous section. Inserting these functions into Propositions 2 and 3 yields the explicit solution as follows:

Corollary 2 *Explicit solution for the example*

a) For $\theta \leq \tilde{\theta}$ an integration equilibrium emerges, where the five unknown endogenous variables V_H, V_L, x_{HH}, x_{HL} and x_{LL} are characterized by the following system of equations:

$$\begin{aligned}
(Ia) \quad rV_H &= y_H + 1 - x_{HH} \\
(IIa) \quad rV_H &= \frac{y_H + y_L}{2} + 1 - x_{HL} \\
(IIIa) \quad rV_H &= y_H + \frac{1}{4}\theta \left[x_{HH}^2 + x_{HL}^2\right] \\
(IVa) \quad rV_L &= y_L + 1 - x_{LL} \\
(Va) \quad rV_L &= y_L + \frac{1}{2}\theta x_{HL} \left[\frac{y_H - y_L}{2} + x_{LL} - \frac{x_{HL}}{2}\right] + \frac{1}{4}\theta x_{LL}^2
\end{aligned}$$

b) For $\theta > \tilde{\theta}$ a segregation equilibrium emerges, where the three unknown endogenous variables V_H, V_L and $\bar{x}$ are characterized by the following system of equations:

$$\begin{aligned}
(Ib) \quad \bar{x} &= \frac{2}{\theta} \cdot (\sqrt{1+\theta} - 1) \\
(IIb) \quad rV_H &= y_H + 1 - \bar{x} \\
(IIIb) \quad rV_L &= y_L + 1 - \bar{x}
\end{aligned}$$

c) The threshold extent $\tilde{\theta}$ is given by

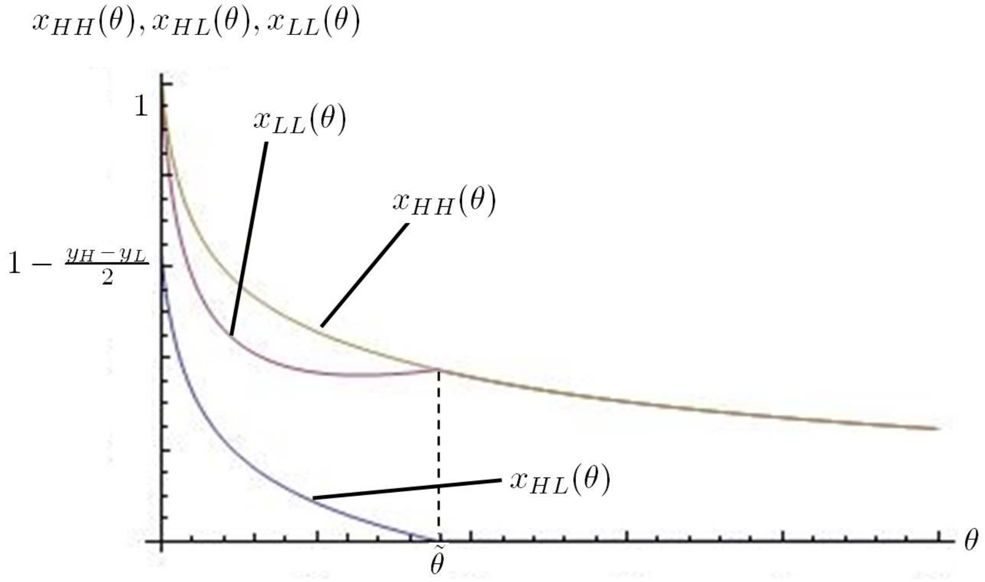
$$\tilde{\theta} = 8\frac{2 - (y_H - y_L)}{(y_H - y_L)^2}. \tag{27}$$

To simplify the discussion further, we choose explicit values for the two income levels and the discount rate and present the model solution graphically. In particular, we use $y_H = 1$ and $y_L = 0.25$. Agents discount future utility with $r = 0.05$.[17]

Figure 3 depicts the reservation qualities $x_{HH}(\theta), x_{HL}(\theta)$ and $x_{LL}(\theta)$ for the specification given above. For a market extent above $\tilde{\theta}$ the solution exhibits complete

[17] We set incomes and the discount rate r to specific values to simplify the example as much as possible. This implies that the figure shows the reservation qualities for different values of α. Since the qualitative findings are independent of specific values of r, we denote all functions as functions depending on θ.

Figure 3: Reservation qualities

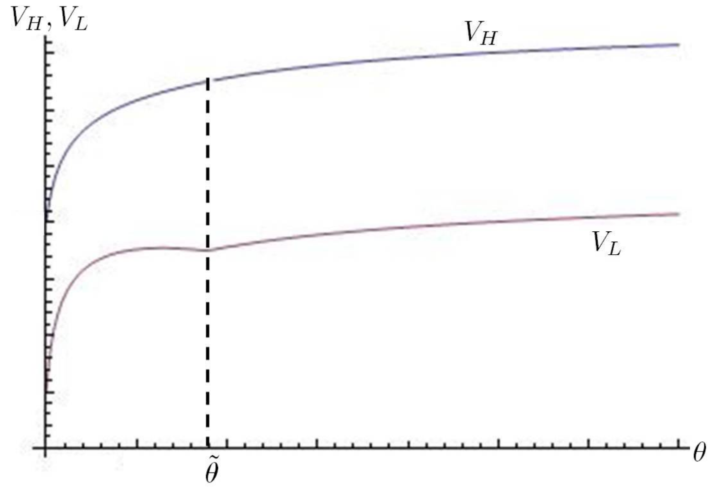


segregation. In this case there are no mixed couples and $x_{HL}(\theta) = 0$. The two reservation qualities for high types within themselves and for low types within themselves coincide as established by Proposition 3. As discussed, there are no mixed couples, and intra-marital redistribution does not play a role. Therefore, agents condition their proposals within their own type only on taste and $\bar{x}$ is identical for both types. In the limit ($\theta \to \infty$) the two reservation qualities $x_{HH}(\theta)$ and $x_{LL}(\theta)$ converge to zero. This is because an infinite market extent implies that all agents meet each other instantaneously.

For $\theta \leq \tilde{\theta}$ integration equilibria emerge. For values around $\tilde{\theta}$, the three reservation qualities clearly exhibit the continuity and differentiability properties discussed in the previous section. What happens if search frictions increase further such that θ approaches zero? First note that equations (IIa) and (IIIa) of Corollary 2 imply $x_{HH} - x_{HL} = \frac{y_H - y_L}{2}$. This distance is the "compensation" in taste required by high type agents from low type agents for the income loss due to redistribution. By the assumptions with respect to $g(\cdot)$, this distance is constant ($\frac{3}{8}$). Hence, in the figure $x_{HL}(\theta)$ and $x_{HH}(\theta)$ are parallel functions for $\theta \leq \tilde{\theta}$.

For maximum search frictions (θ close to zero) the reservation qualities x_{HH} and x_{LL} converge to one. In this case all agents propose to any agent within their own class. This result is driven by the normalization $g(1) = 0$ that states that agents weakly prefer marrying any agent from the same class rather that staying single forever. As discussed, an important feature of the model is that high type agents do not accept all low type agents. In general, the value for $x_{HL}(0)$ can be obtained from part a) of

Figure 4: Lifetime utilities



Proposition 5. In this specific example, we have $x_{HL}(0) = 1 - \frac{y_H - y_L}{2} = \frac{5}{8}$.

A notable feature is that for a certain range of θ the reservation quality within low type agents is *decreasing* in θ. That is, although the probability of meeting other low type agents increases, low agents are less selective within their own class. Intuitively, the decisive high type agents reduce the set of acceptable low type partners if the market extent increases. As a consequence low type agents realize that the *lucky* event of being matched to a high type becomes more unlikely. Hence, in this range they get less choosy regarding low types although market conditions improve.

Figure 4 highlights the consequences of this feature on agents' lifetime utilities. For the aforementioned range of $\theta < \tilde{\theta}$ low type agents would be better off if the extent of the market shrinks slightly. This is driven by the positive probability of being matched with a high type. Low type agents prefer these mixed matches, because they marry a partner with similar taste and a high income. By contrast, high type agents dislike the income reduction through redistribution and would always prefer lower search frictions.

As briefly discussed in the previous section, the continuity properties with respect to agents' lifetime utilities seem to be desirable features. When agents differ only with respect to a vertical characteristic (e.g. Burdett and Coles 1997), they endogenously partition themselves into distinct classes in equilibrium. Within each class all agents obtain the same lifetime utility. Search frictions affect the size of these classes, and agents' experience discrete jumps in their lifetime utility, whenever they drop from one of the classes.

The presence of the second horizontal dimension allows high types to reduce continuously their offers to lower types when search frictions decrease. Low type agents adjust their reaction to this reduction in the offer distribution continuously, and the overall effect for low types' expected utility is the sum of two effects. The first, direct, effect of decreasing search frictions leads to meeting potential partners more often. The second, indirect, effect might cause agents to be more selective when deciding on whom to propose to. The example documents a case when the second effect is larger than the first. For values near $\tilde{\theta}$, the net effect of decreasing search frictions is a reduction of low types' lifetime utility.[18]

5. Extensions and Discussion

In this section we want to show that the model results are robust to relaxing two specific assumptions. The first extension deals with the restriction to two types of agents. We show that the method for solving the model works for any discrete number for the types of agents. Second, we relax the assumption that the payoff with respect to taste is a function of the distance between the two agents' positions. Although it is intuitive and convenient to discuss the payoff in terms of positions on the circle and the resulting distance, it is not necessary.

5.1. More than two types of income

In the previous sections we have analyzed the case of at most two discrete levels of income and horizontal heterogeneity. A central assumption regarding the horizontal dimension is symmetry in payoffs, that is both spouses receive exactly the same noneconomic gain when married (cf. Konrad and Lommerud 2010). This symmetry allows us to partially apply results known from the model involving only a vertical dimension (e.g. Burdett and Coles 1997). In particular, this symmetry implies that the high type agent is decisive for mixed marriages. Hence, in equilibrium all low-type agents who receive offers from a particular high type agent also propose to this agent. This observation is crucial, as it allows to use a step-wise approach to solve for the equilibrium. In the two-type model, low types take the high type's equilibrium behavior essentially as given and perform a conditional maximization. In the following, we will exploit this idea to establish an integration equilibrium for N discrete income levels $y_1 < y_2 < ... < y_N$. In this extended model each type except

[18]Clearly, only agents who are not decisive concerning potentially formed marriages may suffer from decreased search frictions.

the lowest type will be decisive concerning agents with lower vertical traits. Once again, we denote individuals by small letters and income classes with capital letters if a distinction seems helpful. For the following let $J = \{1, 2, ..., N\}$.

Let α_i^J denote the probability that agent i contacts a single of type J who is willing to marry her. Generally, we denote the cumulative probability that the distance of an agent of type J who is willing to marry individual i as $H_{-i}^J(x_J|y_i)$. In general, there are N different types of agents proposing to individual i and hence the lifetime utility is determined by

$$\begin{aligned} V_i = \frac{1}{1+\Delta r} \Bigg[& \Delta f(y_i, -) \\ & + \Delta \left(\sum_J \alpha_i^J \right) \left(\sum_K \frac{\alpha_i^K}{\sum_J \alpha_i^J} E_{-i}^K \left(\max \left\{ V_i, \frac{f(y_i, y_K) + g(x_K)}{r} \right\} \right) \right) \\ & + (1 - \Delta(\sum_J \alpha_i^J))V_i \Bigg] + o(\Delta). \end{aligned} \tag{28}$$

For given offer distributions, we could now reformulate Lemma 2 to describe the reservation utility strategies for the case of N different income levels. However, as the reasoning is identical we proceed directly to equilibrium behavior.

For each type of agent J there is one reservation utility level V_J. Furthermore, each type of agent J has N different reservation qualities concerning partners from every potential type of agent. Hence, there are $N + 1$ unknowns for each type of agent which sum up to $N \cdot (N + 1)$ unknowns to describe individuals' equilibrium strategies completely.

As before matches are formed only by mutual agreements. Exactly as in the two type model, higher types have higher reservation utilities. Hence they are the decisive players when agents of different income levels propose to each other. It follows that the amount of unknowns capturing equilibrium behavior can be reduced to those relevant for the formation of matches. As Type N agents are the highest type in the market, they are always decisive. Therefore, there are N binding reservation quality equations associated with type N agents. We call these "decisiveness equations" in the following. An agent of type $N - 1$ is decisive concerning all lower types and her own type which yields $N - 1$ relevant decisiveness equations. In general, there are J decisiveness equations associated with an agent of type J. Additionally, there are N unknowns concerning the reservation utilities. Thus, the equilibrium is fully described by this set of collectively

$$N + \sum_J J = N + \frac{N \cdot (N+1)}{2}$$

unknowns.

As noted above, we want to proceed directly to the equilibrium solution. For the two-type model, exploiting Lemma 2 provides two equations describing the equilibrium values for the two reservation utilities. Modifying these computations to the present case yields N equations describing the N reservation utilities. Furthermore, each type combination between two individuals of types K and respectively L where $K \geq L$ yields an additional equation of the form

$$rV_K - f(y_K, y_L) = g(\overline{x}_K(y_L)).$$

There are N equations concerning the reservation utilities. For each $K \geq L$ there are K additional equations. Adding up yields

$$N + \sum_J J = N + \frac{N \cdot (N+1)}{2}$$

equations in total which equals the number of unknowns. Hence, this system of equations characterizes the equilibrium solution. Once again, without specifying the distance and payoff functions, no explicit derivations can be offered.

Depending on the parametrization, the resulting equilibrium may exhibit segregation as well as integration outcomes with respect to the N types of agents. Let x_{KL} denote the least acceptable L-type-agent from the perspective of K-type-agents. Hence, whenever a K-type-agent meets an L-type agent of quality x_{KL}, she is indifferent between staying single or marrying this agent. The following proposition summarizes the system of equations for high search frictions in the sense that even the highest (N) type is willing to marry some agents of the lowest type[19]:

Proposition 7 *Complete integration equilibrium*

The solution for the $N + \frac{N \cdot (N+1)}{2}$ *unknown endogenous variables is characterized by a system of two types of equations.*

Decisiveness equations:
$rV_K = f(y_K, y_L) + g(x_{KL})$ *for all* $K \geq L$

[19]Clearly, by the fact that reservation utility is increasing in type and all agents apply reservation utility strategies, this implies that all types of agents are possibly matched in equilibrium.

Reservation utility equations:

$$
\begin{aligned}
rV_K &= f(y_K, -) \\
&\quad + \frac{1}{N}\frac{\alpha}{r}\left[\sum_{L=K+1}^{N} x_{LK}[f(y_K, y_L) + E\left(g(x) \mid x \leq x_{LK}\right) - f(y_K, y_K) - g(x_{KK})\right] \\
&\quad + \frac{1}{N}\frac{\alpha}{r}\sum_{L=1}^{K}\left[\int_0^{x_{KL}} g(x) - g(x_{KL}) dx)\right]
\end{aligned}
$$

In general, depending on the extent of the market, there emerge all intermediate equilibria between the perfect integration equilibria discussed in Proposition 7 and the perfect segregation equilibria. We do not provide solutions for all these intermediate cases. The method to construct these cases is clear from the discussion above, while writing down all possible cases is lengthy and involves bulky notation.

5.2. Different distribution for taste-related payoff

The assumption that the payoff with respect to taste is ex-ante uniformly distributed is restrictive at first glance. We now argue that the present model solution encompasses the analysis for a large range of continuous distribution functions.

Note that the utility gain along the horizontal dimension is interpreted as a result from two steps. First, individuals are distributed uniformly on a circle. Second, agents are randomly matched to each other. In general, there are many ways to define utility depending on the two positions of the spouses; e.g. in a marriage market model with men and women the utility induced by taste could be determined solely through the location of the woman. The central idea of our model is that agents prefer to be matched with agents who are horizontally close to themselves. Therefore, an obvious choice for linking location and the resulting taste-related payoff is the distance x between the two locations. In the model we consider a very general measure of distance, namely $g(x)$.

Since agents are randomly matched to each other and the taste (location) parameter t is uniformly distributed on $[0, 2]$, the distance X of two randomly matched agents is uniformly distributed on $[0, 1]$. However, the solution method and the general form of g allow for a wide range of random variables capturing the taste-related payoff. The following proposition establishes this result:

Proposition 8 *Possible distributions for match-specific values*

Let $X \sim U[0,1]$. Let $H_Z(z)$ denote the cumulative distribution function of a random variable Z where Z has finite support $[a,b] \subset \mathbb{R}$. Choosing a utility specification $g(x) := b - H_Z^{-1}(x)$ for the taste-related utility gain is equivalent to solving the model with $Z \sim H_Z(z)$ and $g(z) = b - z$.

Proposition 8 shows that the assumption of the uniform distribution for the taste-related payoff (distance) is not restrictive. The payoff can follow any continuous probability distribution that has finite support. Hence, there are virtually no restrictions concerning the distribution of match-specific values. As before, the distance $b - z$ is just a normalization which implies that agents are never matched to individuals who would yield a negative payoff along the taste dimension. As in the untransformed model we could easily allow negative payoffs.

6. Conclusion

This article analyzes how matching and sorting takes place when individuals are characterized by both vertical and horizontal ex-ante heterogeneity. Our dynamic model assumes that search frictions are present in the market and that utility is nontransferable. Along the vertical dimension all individuals agree on the ranking of agents and prefer to be matched with the highest agent. This vertical trait can therefore be interpreted as income, wealth or even beauty. Unlike in most of the search-theoretic literature, the horizontal dimension considered here is not ordered. It captures inputs which are not equally ranked across agents. Possible examples are regional preferences concerning a job or sharing hobbies with the spouse. The analysis for our definition of horizontal heterogeneity[20] is quite different from the more common assumption of preferring similar characteristics along an ordered trait like height (e.g. Clark 2007).[21]

One appealing result is that the reservation-utility strategies are continuous in the market extent. By contrast, in the typical vertical model (e.g. Burdett and Coles 1997) the strategies and the corresponding expected lifetime utilities are not continuous. Our result is driven by the fact that agents are able to condition their offers on both dimensions. If the search frictions are large enough, high-type agents start sending offers to low-type agents. When the extent of the market is exactly at

[20] Konrad and Lommerud (2010) employ such a definition of horizontal heterogeneity and analyze the implications of redistributive taxation in a static one-period model.

[21] In these models the probability of being matched is usually not independent of the location. For example, if agents prefer partners of similar height, individuals in the middle of the distribution have higher chances of meeting an appropriate partner.

this threshold, high-type agents propose only to the low-type agent who yields the largest horizontal payoff. When search frictions increase further, high-type agents expand continuously the set of acceptable low-type agents. Low-type agents are decisive only within low-types and choose optimally the range of acceptable payoff with respect to the horizontal trait.

Driven by this continuity property, we can identify ranges of search frictions where the direct (positive) effect of decreasing search frictions is outweighed by the indirect (negative) effect for every type of agent (except the highest). In most cases, agents reduce their set of acceptable partners within their own class when search frictions decrease and hence are better off. However, for low-type agents this relation reverses for certain ranges of search frictions. For these particular ranges the probability for mixed matches is elevated when search frictions increase. If the payoff of low types generated by pooling with the high type is sufficiently high, this effect outweighs the negative effect of meeting less potential partners.

Finally, the approach to tackle the issue of horizontal heterogeneity in the presented way seems appropriate for a handful of reasons. The model is generalizable in many kind of ways. The weights assigned to the utilities from the two traits are not a restriction as long as utility remains additively separable. The payoff derived from the horizontal trait can follow any continuous probability distribution that has finite support. The number of types in the vertical dimension can be increased to an arbitrary natural number.

There are several routes for future research. For simplicity, it is assumed that individuals who leave the market are immediately replaced by clones. Introducing different inflow and outflow specifications should lead to multiple equilibria for a given extent of the market. Another important assumption is the symmetry of the horizontal dimension. This ensures that both agents in a match always receive the same gain along the horizontal dimension. This in turn implies that high-type agents are decisive for mixed matches and considerably simplifies the extension to N discrete income levels.

A. Appendix

Proof of Lemma 1

From equation (2) we know that an agent accepts all proposals which yield her a higher utility than staying single, i.e. agent i accepts if and only if

$$rV_i \leq f(y,y) + g(x). \tag{29}$$

In general, there are two cases to distinguish: In the first case agent i does not accept all proposals which she faces. Thereby, she accepts all proposals up to the taste difference where she is indifferent between marriage and staying single. This determines her reservation quality $\bar{x}_i$ which is then given by

$$rV_i = f(y,y) + g(\bar{x}_i). \tag{30}$$

Using equation (30) we can reformulate the right-hand-side of equation (2). This yields

$$rV_i = f(y,-) + \frac{\alpha_i}{r} \int_0^{\bar{x}_i} \left(g\left(x\right) - g\left(\bar{x}_i\right)\right) dH_{-i}(x).$$

In the second case the agent accepts every proposal she faces. The agent would be even willing to accept agents whose location is beyond the taste difference of $\bar{x}_i$, but these individuals do not propose to her. Formally, we have

$$rV_i < f(y,y) + g(\bar{x}_i) \text{ and } H_{-i}(\bar{x}_i) = 1$$

as every offer is accepted. Furthermore, as the agent always marries when she faces a proposal the maximum operator from equation (2) simplifies and one gets

$$rV_i = f(y,-) + \frac{\alpha_i}{r} E_{-i}\left(f(y,y) + g(x) - rV_i\right).$$

■

Proof of Proposition 1

The derivations of the equations are shown in the text. For the equilibrium property of this characterization we still have to show that the strategies of the agents generate a consistent offer distribution for a particular agent. As every agent decides to propose to all other agents within taste distance $\bar{x}$ by the symmetry of agent's distance to each other all proposals are accepted. If agents calculate this best response on the belief on the offer distributions that all other agents play the strategy described in the proposition, then choices and expectations are consistent with each other. ■

Proof of Corollary 1

The first part simply follows by equations (I) and (II) from Proposition 1 and recalling the utility definition $f(y,-) = f(y,y)$.

For the second part let

$$F(\theta, x(\theta)) := \theta \left[\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x}) \right] - g(\bar{x}).$$

Using the implicit function theorem we establish that $\bar{x}$ is decreasing in extent θ as

$$\begin{aligned} \frac{d\bar{x}}{d\theta} &= -\left(\frac{dF}{d\bar{x}}\right)^{-1} \cdot \frac{dF}{d\theta} \\ &= -\frac{\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x})}{\theta(g(\bar{x}) - g(\bar{x}) - \bar{x}g'(\bar{x})) - g'(x)} \\ &= \frac{\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x})}{g'(x)(1+\theta\bar{x})} < 0. \end{aligned}$$

The last sign holds as the numerator is positive due to $g' < 0$ and the denominator is negative for the same reason. Hence, we get $d\bar{x}/d\theta < 0$, i.e. the reservation quality required by the agents is increasing in the extent of the market.

It remains to show the limiting behavior of $\bar{x}$. Letting $\theta \to 0$ and finding a solution to equation (10) is equivalent to solving $g(\bar{x}) = 0$ which yields $\bar{x} = 1$. The considerations for $\theta \to \infty$ are similar. We have to find $\bar{x}$ which fulfills

$$g(\bar{x}) = \theta \left[\int_0^{\bar{x}} g(x)dx - \bar{x}g(\bar{x}) \right].$$

For given θ we can choose $\bar{x}$ arbitrarily close to 0 such that the right-hand-side is arbitrarily close to zero, whereas the left-hand-side still yields a finite value. Hence, this equation has always a solution. Furthermore, as the right-hand-side is increasing in θ, we have that $\lim_{\theta\to\infty} \bar{x}(\theta) = 0$. ■

Proof of Lemma 2

This is a direct generalization of the proof from Lemma 1. The rest follows directly from the text. ■

Proof of Proposition 2

The proof follows directly from the text. ■

Proof of Proposition 3

The proof follows directly from the text and from Proposition 1. ■

Proof of Proposition 4

First note that $x_{HL}(\tilde{\theta}) = 0$, which we will use at some points throughout the proof.

First, we consider the continuity of $x_{HH}(\theta)$. The continuity for $\theta \in \mathbb{R}_+ \setminus \{\tilde{\theta}\}$ can be seen directly from the equation system (I)-(V) of the integration equilibrium given in Proposition 2 and the expression $g(\bar{x})$ from the income segregation equilibrium given in Proposition 3. For the case of $\theta = \tilde{\theta}$ equating (I) and (III) together with $x_{HL}(\tilde{\theta}) = 0$ yields

$$g(x_{HH}) = \frac{\theta}{2}\left[\int_0^{x_{HH}} g(x)dx - x_{HH}g(x_{HH})\right].$$

Hence, this solution for x_{HH} at $\tilde{\theta}$ coincides with the solution of $g(\bar{x})$ of the segregation equilibrium. Since the solution is the same on both sides of $\tilde{\theta}$ we have continuity for x_{HH} for all $\theta \in \mathbb{R}_+$.

Now we analyze $x_{LL}(\theta)$. The continuity for $\theta \in \mathbb{R}_+ \setminus \{\tilde{\theta}\}$ can be seen directly from the equation system (I)-(V) of the integration equilibrium given in Proposition 2 and the expression $g(\bar{x})$ from the income segregation equilibrium given in Proposition 3. For the case of $\theta = \tilde{\theta}$ equating (IV) and (V) together with $x_{HL}(\tilde{\theta}) = 0$ yields

$$g(x_{LL}) = \frac{\theta}{2}\left[\int_0^{x_{LL}} g(x)dx - x_{LL}g(x_{LL})\right].$$

Hence, this solution for x_{HL} at $\tilde{\theta}$ coincides with the solution of $g(\bar{x})$ of the segregation equilibrium. Since the solution is the same on both sides of $\tilde{\theta}$ we have continuity for x_{LL} for all $\theta \in \mathbb{R}_+$.

Finally, we analyze $x_{HL}(\theta)$. The continuity of x_{HL} for all $\theta \in (0, \tilde{\theta})$ follows directly from the equation system (I)-(V). For $\theta > \tilde{\theta}$ we have $x_{HL} \equiv 0$. Hence, as x_{HL} is zero per definition at $\tilde{\theta}$ we have continuity for all $\theta \in \mathbb{R}_+$. ■

Proof of Proposition 5

a) Let $\theta = 0$. By equations (I) and (III) from Proposition 2 we get $g(x_{HH}) = 0$. By the definition of g this yields $\lim_{\theta \to 0} x_{HH}(\theta) = 1$. The same argument for equations (IV) and (V) from Proposition 2 shows $\lim_{\theta \to 0} x_{LL}(\theta) = 1$. Furthermore, by equating (I) and (II) from Proposition 2 one gets that the difference between least acceptable partners (measured in g) is a constant:

$$g(x_{HL}) - g(x_{HH}) = f(y_H, y_H) - f(y_H, y_L) \tag{31}$$

We already know for $\theta \to 0$ we have $g(x_{HH}) = 0$. Using this in equation (31) shows the last claim from part (a).

b) For $\theta \to \infty$ we have complete segregation and hence Corollary 1 applies. This completes the proof. ■

Proof of Proposition 6

By Proposition 2 and Proposition 3 the differentiability of reservation qualities x_{LL}, x_{HH} and x_{HL} for all $\theta \in \mathbb{R}_+ \setminus \tilde{\theta}$ is obvious. The only thing which remains to be shown is the differentiability of x_{HH} in $\tilde{\theta}$.

For reasons of simplicity we analyze the system of equations given in Proposition 2 and Proposition 3 as functions of the market extent α instead of θ; the latter definition of market extent is just a transformation by the constant discount factor r. By the implicit functions theorem the left- and the right-hand-side derivatives of the reservation quality of the high type among her own class determine as

$$\lim_{\Theta \nearrow \tilde{\Theta}} \frac{\partial x_{HH}}{\partial \alpha} = \lim_{\Theta \searrow \tilde{\Theta}} \frac{\partial x_{HH}}{\partial \alpha} = \frac{\int_0^{x_{HH}} g(x)dx - x_{HH} g(x_{HH})}{g'(x_{HH})(2r + x_{HH}\alpha)}.$$

Hence, x_{HH} is differentiable in $\mathbb{R}_+$. Furthermore, the analysis of the derivatives shows easily, that the reservation qualities x_{HH} and x_{HL} are decreasing resp. non-increasing everywhere. ■

Proof of Corollary 2

All stated equations are the results of calculations using the formula given in Proposition 2 for part a) and Proposition 3 for part b).

It only remains to calculate threshold extent $\tilde{\theta}$. From equation (22) it is clear that at $\tilde{\theta}$ we have

$$y_H + 1 - x_{HH} = \frac{y_H + y_L}{2} + 1 \Leftrightarrow x_{HH} = \frac{y_H - y_L}{2}.$$

By Proposition 4 we know that $x_{HH}(\theta)$ is continuous at $\tilde{\theta}$ and hence, using equation (Ib) at $\tilde{\theta}$ it must hold that

$$\frac{y_H - y_L}{2} = \frac{2}{\tilde{\theta}}(\sqrt{1+\tilde{\theta}} - 1).$$

Solving this for $\tilde{\theta}$ yields

$$\tilde{\theta} = 8\frac{2-(y_H - y_L)}{(y_H - y_L)^2}.$$

This completes the proof. ■

Proof of Proposition 8

So far, distance X is uniformly distributed on $[0, 1]$. Therefore, we have $F(x) = P(X \leq x) = x$. Let Z be another random variable with finite support $[a, b] \subset \mathbb{R}$ with $H_Z(z) = P(Z \leq z)$ denoting the corresponding cumulative distribution function and associated quantile function $H_Z^{-1}(q)$. Let Y denote the random variable obtained from transforming X according to $Y := H_Z^{-1}(X)$.

For the distribution of Y we then have:

$$\begin{aligned} P(Y \leq y) &= P(H_Z^{-1}(X) \leq y) = P(H_Z(H_Z^{-1}(X)) \leq H_Z(y)) \\ &= P(X \leq H_Z(y)) = H_Z(y). \end{aligned}$$

Hence, the associated cumulative distribution to random variable Y is function H_Z.

We transform the distance as follows with function $g(x) = b - H^{-1}(x)$. Solving this model is the same as solving the previous untransformed model with taste payoff $g(z) = b - z$ and $Z \sim H_Z(z)$. ■

Proof of Proposition 7

The proof follows directly from the text. ■

References

Adachi, H. (2003). A search model of two-sided matching under nontransferable utility, *Journal Of Economic Theory* 113(2): 182–198.

Atakan, A. E. (2006). Assortative matching with explicit search costs, *Econometrica* 74(3): 667–680.

Becker, G. S. (1973). A theory of marriage: Part i, *The Journal of Political Economy* 81(4): 813–846.

Becker, G. S. (1974). A theory of marriage: Part ii, *The Journal of Political Economy* 82(2): 11–26.

Bloch, F. and Ryder, H. (2000). Two-sided search, marriages, and matchmakers, *International Economic Review* 41(1): 93–115.

Burdett, K. and Coles, M. G. (1997). Marriage and class, *Quarterly Journal Of Economics* 112(1): 141–168.

Burdett, K. and Coles, M. G. (1999). Long-term partnership formation: Marriage and employment, *Economic Journal* 109(456): F307–F334.

Clark, S. (2007). Matching and sorting when like attracts like, ESE Discussion Papers 171, Edinburgh School of Economics, University of Edinburgh.

Eeckhout, J. (1999). Bilateral search and vertical heterogeneity, *International Economic Review* 40(4): 869–887.

Kalmijn, M. (2006). Educational inequality and family relationships: Influences on contact and proximity, *European Sociological Review* 22(1): 1–16.

Kalmijn, M. and van Tubergen, F. (2006). Ethnic intermarriage in the netherlands: confirmations and refutations of accepted insights, *European Journal Of Population-Revue Europeenne De Demographie* 22(4): 371–397.

Konrad, K. A. and Lommerud, K. E. (2010). Love and taxes - and matching institutions, *Canadian Journal of Economics* 43(3): 919–940.

Lundberg, S. (2011). Psychology and family economics, *Perspektiven der Wirtschaftspolitik* 12: 66–81.

Lundberg, S. and Pollak, R. A. (2007). The american family and family economics, *Journal of Economic Perspectives* 21(2): 3–26.

Mare, R. D. (1991). Five decades of educational assortative mating, *American Sociological Review* 56(1): 15–32.

Salop, S. C. (1979). Monopolistic competition with outside goods, *The Bell Journal of Economics* 10(1): pp. 141–156.

Schmalensee, R. (1978). Entry deterrence in the ready-to-eat breakfast cereal industry, *The Bell Journal of Economics* 9(2): pp. 305–327.

Shimer, R. and Smith, L. (2000). Assortative matching and search, *Econometrica* 68(2): 343–369.

Smith, L. (2006). The marriage model with search frictions, *Journal Of Political Economy* 114(6): 1124–1144.