Investments in Large Bilateral Matching Markets Revisited

Deniz Dizdar*

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Abstract

We analyze large bilateral matching markets in which agents from both sides must decide on costly productive investments into attributes before they match and bargain with potential partners. Following the pioneering work of Cole, Mailath and Postlewaite (2001) (henceforth (CMP)), we assume a frictionless matching market with transferable utility. Unlike all related literature on investment prior to two-sided matching that we are aware of (including the work on non-transferable utility and on matching with search frictions), we allow for very general investment decisions (in particular multidimensional ones) and match values, as well as for general heterogeneity with respect to agents' investment costs. Using insights about structural properties of the outcomes of frictionless two-sided matching markets with transferable utility that have been obtained in the mathematical theory of optimal transport and in the work of Gretzky, Ostroy and Zame (1992, 1999), we are able to provide a rigorous notion of investment equilibrium, the analog of (CMP)'s ex-post contracting equilibrium. In an investment equilibrium, no agent has an incentive to deviate in his investment, given others' equilibrium investment decisions and the equilibrium outcome of the resulting competitive matching market. We show that ex-ante efficient (welfare-maximizing) investments can always be sustained in investment equilibrium. This proves that a main result of (CMP) holds far more generally

*Bonn Graduate School of Economics

and does not hinge on restrictive structural assumptions such as one-dimensional type spaces and supermodular technology, which were needed for the original proof of (CMP). We also briefly review the one-dimensional supermodular model and identify an interesting inefficient equilibrium that differs substantially from the examples of (CMP). In our example, underinvestment (by "lower medium" types) and overinvestment (by "upper medium" types) may occur simultaneously, so that the economy lacks an efficient middle sector. Moreover, this type of inefficiency need not unravel even for very heterogenous economies. Still, potential inefficiencies in the "1-d supermodular world" are special (see sections 4 and 5). An analysis of possible inefficiencies in more general cases is work in progress, and will feature the next version of this working paper.

1 Model

1.1 Fundamentals: Continuum Populations, Investment Costs and Match Value

There are two continuum populations of agents. Following (CMP), we will refer to these as buyers and sellers. We assume throughout that all agents have quasi-linear utility functions and that utility is transferable via transfer payments. Both buyers and sellers are heterogenous with respect to their cost of investment. A buyer of type $b \in B$ who makes a productive investment into an attribute $x \in X$ incurs a cost $c(x, b)$. Similarly, a seller of type $s \in S$ can invest into an attribute $y \in Y$ at cost $d(y, s)$. We assume that B, S, X and Y are compact metric spaces[1] (we suppress the metrics and induced topologies in the notation), and that $c : X \times B \to \mathbb{R}_+$ and $d : Y \times S \to \mathbb{R}_+$ are continuous functions.

Agents' investments into attributes determine how much value they can create in a relationship with an agent from the other side of the market. If a buyer with attribute

[1]The theory of optimal transport that we use is developed for general Polish spaces, and many of our results could in principle be obtained in that setting. However, this gain in generality would be more than offset by a loss in clarity due to further technical assumptions. Thus we stick to compact metric type- and attribute spaces.

x and a seller with attribute y match, they generate a match value $v(x, y)$. We assume that the match value function $v : X \times Y \to \mathbb{R}_+$ is continuous. Agents who remain unmatched obtain value zero. This latter assumption is made merely for simplicity. A more general model in which it may be a socially valuable option to leave some agents unmatched even though there are still potential partners available is ultimately equivalent.

The heterogenous continuum populations of buyers and sellers are described by Borel probability measures[2] μ on B and ν on S. In fact, all measures that occur in this paper are Borel probability measures. The "generic" case is of course the one in which there is a long side and a short side of the market (that is, there are more buyers than sellers or vice versa). However, this can easily be incorporated by introducing (isolated) "void" types on the short side. Such void types $b_\varnothing \in B$ and $s_\varnothing \in S$ always choose void investments $x_\varnothing \in X$ and $y_\varnothing \in Y$ at zero cost. To avoid bunching real agents with, nonexistent, void agents then, one just has to define the void investments to be sufficiently costly for all $b \neq b_\varnothing$, $s \neq s_\varnothing$, so that no real agent ever chooses them. In accordance with our assumption that unmatched agents create no value, we set $v(x_\varnothing, \cdot) \equiv 0$ and $v(\cdot, y_\varnothing) \equiv 0$.

The timing structure is the same as in (CMP). First, all agents invest simultaneously and non-cooperatively. Then, matches are formed in a bilateral matching market with no search frictions, and value is shared according to the market outcome. The purpose of the two following sections, 1.2 and 1.3, is to show that this informal description can be made precise in our general setting, and thus way beyond the one-dimensional, supermodular and smooth model of (CMP). In fact, a basic version of a "state of the art" duality theorem on the structure of solutions of optimal transport problems can be used to obtain a transparent characterization of outcomes of the attribute market, and consequently also an unambiguous formulation of agents' investment problems, and of "investment equilibria".

[2]We choose to work with normalized measures which is common in and suited for the theory of optimal transport. Gretzky, Ostroy and Zame (1992, 1999) use non-negative Borel measures, which is useful for discussing properties of the "social gains function" that plays a key role in their analysis.

1.2 The Transferable Utility Assignment Model

We model the two-sided economy that results from the simultaneous investments into attributes as a bilateral matching market with no search frictions. More precisely, we employ the continuum version of the transferable utility assignment model, which has been analyzed by Gretzky, Ostroy and Zame (1992, 1999) (henceforth (GOZ1) and (GOZ2)) as a generalization of the Shapley and Shubik (1972) model.

The basic data of the transferable utility assignment model are two population measures of attributes, $\widetilde{\mu}$ on X and $\widetilde{\nu}$ on Y, and the match value function v. Since v is continuous, the framework is equivalent to the one used in (GOZ2) and in section 3.5 of (GOZ1). The transferable utility assignment model can be analyzed in three different ways: as a linear programming problem, as a cooperative game, and as an exchange economy. Extending the results of Shapley and Shubik (1972) from finite to continuum economies, (GOZ1) proved that these three views are in fact equivalent. In their words, "solutions to the linear programming problem (of maximizing aggregate value) are Walrasian allocations, and solutions to the dual linear programming problem are core utilities and correspond to Walrasian prices" (this concise formulation is taken from GOZ2). Both (GOZ1) and (GOZ2) are excellent references for details on these equivalences. In sum, they permit us to focus only on the linear program and its dual.

In this context, we apply a recent formulation of the fundamental Kantorovich duality theorem of the theory of optimal transport (Proposition 1 below, which is adapted from Theorem 5.10 in Villani (2009)). This theorem conveys a very clear picture of the structure of solutions of the transferable utility assignment model, and in particular of the relationship between primal and dual solutions. For our purposes, its two most important consequences will be the following. First, those core utilities (i.e. equilibrium attribute rents, Walrasian prices) which adequately reflect frictionless matching and bargaining for everybody are pinned down for **all** existing attributes, not just almost surely. This is crucial for analyzing agents' investment decisions later on. Second, the theorem renders worries about a suitable definition of feasibility of attribute rents unnecessary[3].

[3]Such worries are discussed at some length by Cole, Mailath and Postlewaite (2001).

We next describe the linear program as well as the additional concepts needed to understand the statement of Proposition 1. Our exposition is brief since it is merely a reproduction of material from Chapter 5 in Villani (2009), and parts of it also occur here and there in (GOZ1) and (GOZ2).

The primal linear program is the problem of finding an allocation (matching of attributes) that maximizes aggregate value. The feasible allocations are the couplings of $\widetilde{\mu}$ and $\widetilde{\nu}$, that is the measures $\widetilde{\pi}$ on $X \times Y$ with marginals $\widetilde{\mu}$ and $\widetilde{\nu}$[4]. We write $\Pi(\widetilde{\mu}, \widetilde{\nu})$ for the set of all these couplings. Thus, the linear program of maximizing aggregate value is to find a $\widetilde{\pi} \in \Pi(\widetilde{\mu}, \widetilde{\nu})$ that attains

$$\sup_{\widetilde{\pi}' \in \Pi(\widetilde{\mu}, \widetilde{\nu})} \int_{X \times Y} v \, d\widetilde{\pi}'. \tag{1}$$

The dual linear program reads as follows. Find functions $\widetilde{\psi} \in L^1(\widetilde{\mu})$ and $\widetilde{\phi} \in L^1(\widetilde{\nu})$ from the constraint set specified below (where the constraint qualification must hold for a pair of representatives from the L^1-equivalence classes) which attain

$$\inf_{\{(\widetilde{\psi}', \widetilde{\phi}') \in L^1(\widetilde{\mu}) \times L^1(\widetilde{\nu}) \mid \, \widetilde{\phi}'(y) + \widetilde{\psi}'(x) \geq v(x,y) \text{ for all } (x,y) \in X \times Y\}} \left(\int_Y \widetilde{\phi}' \, d\widetilde{\nu} + \int_X \widetilde{\psi}' \, d\widetilde{\mu} \right).$$

Villani's Theorems 4.1 and 5.10, or alternatively Theorems 1 and 2 of (GOZ1), imply that both the primal and the dual problem have solutions, and that the optimal values coincide.

From the equality of optimal values, it follows that for **any** optimal coupling $\widetilde{\pi}$ and **any** optimal $(\widetilde{\psi}, \widetilde{\phi})$, $\widetilde{\phi}(y) + \widetilde{\psi}(x) = v(x, y)$ must hold $\widetilde{\pi}$- almost surely. However, this almost sure property of core utilities is not sufficient for our purposes. First, to analyze investment decisions we need to specify an unambiguous equilibrium rent for every attribute. Second, we want that the rents constitute a feasible sharing of the match value for all formed matches, not just almost surely. Proposition 1 attains these two desiderata in a particularly elegant way.

The measure supports $\mathrm{Supp}(\widetilde{\mu})$ and $\mathrm{Supp}(\widetilde{\nu})$ describe the sets of existing attributes, that is those attributes for which there are infinitesimal agents with that attribute.

[4]Note that by our assumptions that match value is non-negative and that unmatched agents create zero value, we need not explicitly consider the possibility that agents remain unmatched. Those agents who match with a void type are of course de facto unmatched.

Now, if rents for seller attributes are $\widetilde{\phi}(y)$ for $y \in \text{Supp}(\widetilde{\nu})$, then a buyer with attribute $x \in \text{Supp}(\widetilde{\mu})$ can get a market return of

$$\widetilde{\psi}(x) = \sup_{y \in \text{Supp}(\widetilde{\nu})} \left(-\widetilde{\phi}(y) + v(x,y) \right).$$

Similarly for sellers,

$$\widetilde{\phi}(y) = \sup_{x \in \text{Supp}(\widetilde{\mu})} \left(-\widetilde{\psi}(x) + v(x,y) \right).$$

So, in the terminology of optimal transport, **with respect to the sets** $\text{Supp}(\widetilde{\mu})$ and $\text{Supp}(\widetilde{\nu})$ (which are compact metric spaces here), $\widetilde{\psi}$ should be a $(-v)$-convex function, and $-\widetilde{\phi}$ should be its $(-v)$-transform, $-\widetilde{\phi}(y) = \inf_{x \in \text{Supp}(\widetilde{\mu})} \left(\widetilde{\psi}(x) - v(x,y) \right) =: \widetilde{\psi}^{-v}(y)$. We refer the reader to Chapter 5 of Villani (2009) for basic general facts about $(-v)$-convexity and $(-v)$-transforms, in particular to the general Definition 5.2 and to Proposition 5.8, which says that a function is $(-v)$-convex (with respect to the given sets) if and only if it coincides with its second transform. We have chosen signs so as to facilitate comparison with Villani's text, which works with the "cost" $-v$ rather than with the value v. Villani's Theorem 5.10, applied for a continuous match value function, then immediately implies the following proposition.

Proposition 1 *It holds*

$$\max_{\widetilde{\pi} \in \Pi(\widetilde{\mu},\widetilde{\nu})} \int_{X \times Y} v \, d\widetilde{\pi} \; = \min_{\{\widetilde{\psi} | \widetilde{\psi} \, is \, (-v)-convex \; w.r.t. \, \text{Supp}(\widetilde{\mu}) \; and \, \text{Supp}(\widetilde{\nu})\}} \left(-\int_Y \widetilde{\psi}^{-v} \, d\widetilde{\nu} + \int_X \widetilde{\psi} \, d\widetilde{\mu} \right).$$

Moreover, there is a closed set $\widetilde{\Gamma} \subset \text{Supp}(\widetilde{\mu}) \times \text{Supp}(\widetilde{\nu})$ *such that*

$$\begin{cases} \widetilde{\pi} \textit{ is optimal in the primal problem if and only if } \text{Supp}(\widetilde{\pi}) \subset \widetilde{\Gamma}, \\ \textit{a (-v)-convex } \widetilde{\psi} \textit{ is optimal in the dual problem if and only if } \widetilde{\Gamma} \subset \partial_{-v}\widetilde{\psi}, \end{cases}$$

where the $(-v)$*- subdifferential of* $\widetilde{\psi}$*,* $\partial_{-v}\widetilde{\psi}$ *is defined as*

$$\partial_{-v}\widetilde{\psi} := \left\{ (x,y) \in \text{Supp}(\widetilde{\mu}) \times \text{Supp}(\widetilde{\nu}) | -\widetilde{\psi}^{-v}(y) + \widetilde{\psi}(x) \; = \; v(x,y) \right\}. \tag{2}$$

We thus make the following definition.

Definition 1 *A **stable and feasible bargaining outcome** for the assignment economy* $(\widetilde{\mu}, \widetilde{\nu}, v)$ *is a pair* $(\widetilde{\pi}, \widetilde{\psi})$*, such that* $\widetilde{\pi} \in \Pi(\widetilde{\mu}, \widetilde{\nu})$ *is an optimal solution for the primal Kantorovich problem and such that the function* $\widetilde{\psi}$*,* $(-v)$*-convex with respect to Supp*$(\widetilde{\mu})$ *and Supp*$(\widetilde{\nu})$*, is an optimal solution for the dual Kantorovich problem.*

Proposition 1 ensures the existence of a stable and feasible bargaining outcome. For any such solution the rents (core utilities, Walrasian prices) are pinned down for all existing attributes. In addition, since $-\widetilde{\psi}^{-v}(y) + \widetilde{\psi}(x) = v(x, y)$ holds on Supp$(\widetilde{\pi})$, the attribute rents constitute a feasible sharing of match value for all formed matches.

Remark 1 *The $(-v)$-subdifferential of a $(-v)$-convex function as defined in* (2) *is a $(-v)$-cyclically monotone set. That is, for all $K \in \mathbb{N}$, $(x_1, y_1), ..., (x_K, y_K) \in \partial_{-v}\widetilde{\psi}$, and $y_{K+1} = y_1$ it holds,*

$$\sum_{i=1}^{K} v(x_i, y_i) \geq \sum_{i=1}^{K} v(x_i, y_{i+1}).$$

Hence, Proposition 1 implies that the support of any optimal coupling is a $(-v)$-cyclically monotone set. This should sound familiar for readers with a background in multidimensional mechanism design.

Finally, continuity of v and compactness of Supp$(\widetilde{\mu})$ and Supp$(\widetilde{\nu})$ imply that **any** $(-v)$-convex function with respect to these sets, $\widetilde{\eta}$ say, is **continuous** and so is its $(-v)$-transform. The proof for this is straightforward. It also follows immediately from the proof of Theorem 6 in (GOZ1) who in fact "$(-v)$-convexify" a given dual solution to extract a continuous representative in the same L^1-equivalence class. In particular, $\partial_{-v}\widetilde{\eta}$ is closed.

1.3 Investment Equilibria

With the description of possible market outcomes for given attribute populations (Definition 1) at hand, we may now turn to agents' non-cooperative investment decisions. The notion of investment equilibrium that we employ is fully analogous to the ex-post contracting equilibrium of Cole, Mailath and Postlewaite. Informally speaking, an investment equilibrium consists of investment decisions for all agents and of a stable and feasible bargaining outcome for the resulting assignment economy, such that nobody has an incentive to deviate at the investment stage.

The continuum model requires reasonable solutions for two technical issues related to describing candidates for equilibrium investments. These problems occur already for the special case analyzed in (CMP). First, aggregate investments must lead to populations of attributes that can be described by Borel measures. The leading case is the one in which investments are given by measurable functions $\beta : B \to X$ and $\sigma : S \to Y$.

In fact, Cole, Mailath and Postlewaite consider a special case of this. In their model, $B = S = [0, 1]$, $\mu = \nu = U[0, 1]$, $X = Y = \mathbb{R}_+$ and β and σ are "well-behaved". That is, β and σ are strictly increasing with finitely many discontinuities, Lipschitz on intervals of continuity points, and have no isolated values.

The last one of these regularity properties is related to the second technical issue, which we will discuss below.

In general, it may be that agents of the same type make different investments because they plan to match with different types of partners. For instance, this must occur quite generally (in equilibrium) whenever the type distributions have atoms. Such cases may also be accommodated: the induced attribute populations are Borel measures when agents have measurable investment functions $\beta : B \times S \to X$ and $\sigma : B \times S \to Y$ which **explicitly** depend on the type of agent from the other side that one "plans" to match with, provided that there is a "pre-assignment" $\pi \in \Pi(\mu, \nu)$ of buyers and sellers. Indeed, the push-forwards (image measures) $\beta_{\#}\pi$ and $\sigma_{\#}\pi$ are Borel measures in this case. Measurable investment functions that depend only on one type can of course be interpreted as a special case: if $\hat{\beta} : B \to X$ is measurable, then $\beta(b, s) := \hat{\beta}(b)$ is product measurable, and $\beta_{\#}\pi = \hat{\beta}_{\#}\mu$ for **any** coupling $\pi \in \Pi(\mu, \nu)$.

The next lemma asserts that we may use the more general setting (measurable $\beta : B \times S \to X$, $\sigma : B \times S \to Y$ along with a $\pi \in \Pi(\mu, \nu)$) without "forgetting" infinitesimal individuals from $\mathrm{Supp}(\mu)$ and $\mathrm{Supp}(\nu)$, the sets of existing agents.

Lemma 1 *Let $P_B(b, s) = b$ and $P_S(b, s) = s$ be the coordinate projections. For any $\pi \in \Pi(\mu, \nu)$ it holds*

$$\mathrm{Supp}(\mu) \;=\; \mathrm{P}_B(\mathrm{Supp}(\pi)) \;\; \textit{and} \;\; \mathrm{Supp}(\nu) \;=\; \mathrm{P}_S(\mathrm{Supp}(\pi)).$$

Proof. See Appendix. ■

We call (β, σ, π) an **investment profile** and next formulate the regularity requirement that is analogous to the "no isolated points" condition in (CMP).

Definition 2 *An investment profile* (β, σ, π) *is said to be* ***regular*** *if it holds for all* $(b, s) \in \mathrm{Supp}(\pi)$ *that* $\beta(b, s) \in \mathrm{Supp}(\beta_{\#}\pi)$ *and* $\sigma(b, s) \in \mathrm{Supp}(\sigma_{\#}\pi)$*.*

Regularity of an investment profile ensures that there are no buyers and no sellers whose attributes get lost in the description $(\beta_{\#}\pi, \sigma_{\#}\pi, v)$ of the attribute assignment economy. In addition, as a consequence of a general property of image measures we have:

Lemma 2 *Let* (β, σ, π) *be a regular investment profile. Then* $\beta(\mathrm{Supp}(\pi))$ *is dense in* $\mathrm{Supp}(\beta_{\#}\pi)$*,* $\sigma(\mathrm{Supp}(\pi))$ *is dense in* $\mathrm{Supp}(\sigma_{\#}\pi)$*, and* $(\beta, \sigma)(\mathrm{Supp}(\pi))$ *is dense in* $\mathrm{Supp}((\beta, \sigma)_{\#}\pi)$*.*

Proof. See Appendix. ■

So, for a regular investment profile, $\beta(\mathrm{Supp}(\pi))$ $(\sigma(\mathrm{Supp}(\pi)))$ is contained and dense in $\mathrm{Supp}(\beta_{\#}\pi)$ $(\mathrm{Supp}(\sigma_{\#}\pi))$. Naturally, $\beta(\mathrm{Supp}(\pi))$ and $\sigma(\mathrm{Supp}(\pi))$ need not be closed or even merely measurable in general, but this does not cause serious problems - neither for the definition of a stable and feasible bargaining outcome (see the next technical lemma) nor for formulating agents' investment problems (see below).

Lemma 3 *Let* (β, σ, π) *be a regular investment profile. Let* $\widetilde{\psi} : \beta(\mathrm{Supp}(\pi)) \to \mathbb{R}$ *be* $(-v)$*-convex with respect to the (generally not closed) sets* $\beta(\mathrm{Supp}(\pi))$ *and* $\sigma(\mathrm{Supp}(\pi))$*, let* $\widetilde{\psi}^{-v}$ *be its transform, and let* $\widetilde{\pi} \in \Pi(\beta_{\#}\pi, \sigma_{\#}\pi)$ *be such that* $-\widetilde{\psi}^{-v}(y) + \widetilde{\psi}(x) = v(x, y)$ *on a dense subset of* $\mathrm{Supp}(\widetilde{\pi})$*. Then there is a unique extension of* $(\widetilde{\psi}, \widetilde{\psi}^{-v})$ *to a* $(-v)$*-dual pair with respect to the compact metric spaces* $\mathrm{Supp}(\beta_{\#}\pi)$ *and* $\mathrm{Supp}(\sigma_{\#}\pi)$*, and with this extension* $(\widetilde{\psi}, \widetilde{\pi})$ *becomes a stable and feasible bargaining outcome in the sense of Definition 1.*

Proof. See Appendix. ■

We still have to make precise how an agent who deviates with his investment affects both his own rent and the rents of other agents. We follow Cole, Mailath and Postlewaite here and assume that a deviation by a single (infinitesimal, zero mass) agent does not affect the rent of any other agent. This is justified formally (the induced attribute population measures remain unchanged). In addition, regularity implies a justification that does not rely on simply ignoring a deviation on a set of measure zero (which may be implausible in interpretations where one takes the pre-assignment seriously). In a regular investment profile, no agent's investment is isolated: every neighborhood of the investment that the agent makes according to the profile has strictly positive mass. So, by continuity of v, whoever was meant to trade with a deviating agent has infinitely many "equivalent" outside options for bargaining.

Given that the stable and feasible bargaining outcome for $(\beta_{\#}\pi, \sigma_{\#}\pi, v)$ is $(\widetilde{\pi}, \widetilde{\psi})$, a buyer who chooses attribute $x \in X$ (for a deviation this may of course be an isolated attribute) can get a gross rent of

$$\widetilde{r}_B(x) \;=\; \sup_{y \in \mathrm{Supp}(\sigma_{\#}\pi)} \Big(v(x,y) + \widetilde{\psi}^{-v}(y) \Big) \,.$$

The continuity of $\widetilde{\psi}^{-v}$ implies that it is irrelevant whether the supremum is taken over $\sigma(\mathrm{Supp}(\pi))$ or its completion $\mathrm{Supp}(\sigma_{\#}\pi)$. In addition, $\widetilde{r}_B$ is continuous (by the Maximum Theorem) and, of course, $\widetilde{r}_B(x) \;=\; \widetilde{\psi}(x)$ for all $x \in \mathrm{Supp}(\beta_{\#}\pi)$. Similarly, sellers with attribute $y \in Y$ obtain

$$\widetilde{r}_S(y) \;=\; \sup_{x \in \mathrm{Supp}(\beta_{\#}\pi)} \Big(v(x,y) - \widetilde{\psi}(x) \Big) \,,$$

which coincides with $-\widetilde{\psi}^{-v}(y)$ on $\mathrm{Supp}(\sigma_{\#}\pi)$.

Definition 3 *An* ***investment equilibrium*** *is a tuple* $((\beta, \sigma, \pi), (\widetilde{\pi}, \widetilde{\psi}))$*, where* (β, σ, π) *is a regular investment profile and* $(\widetilde{\pi}, \widetilde{\psi})$ *is a stable and feasible bargaining outcome for* $(\beta_{\#}\pi, \sigma_{\#}\pi, v)$*, such that it holds for all* $(b, s) \in \mathrm{Supp}(\pi)$ *that*

$$\widetilde{\psi}(\beta(b,s)) - c(\beta(b,s), b) \;=\; \sup_{x \in X} \left(\widetilde{r}_B(x) - c(x,b) \right) \;=:\; r_B(b),$$

and

$$-\widetilde{\psi}^{-v}(\sigma(b,s)) - d(\sigma(b,s), s) \;=\; \sup_{y \in Y} \left(\widetilde{r}_S(y) - d(y,s) \right) \;=:\; r_S(s).$$

Note that, by the Maximum Theorem again, the net rent functions r_B and r_S are continuous.

2 Ex-Ante Optimal Solutions

The **net match value** that agents b and s can achieve when they make jointly optimal investments is

$$\max_{x \in X, y \in Y} \left(v(x,y) - c(x,b) - d(y,s) \right). \tag{3}$$

Since X and Y are compact and since v, c and d are continuous, jointly optimal investments $(x^*(b,s), y^*(b,s))$ exist for all $(b,s) \in B \times S$. In general, these solutions need of course not be unique. We define the ex-ante or net match value function as

$$w(b,s) := v(x^*(b,s), y^*(b,s)) - c(x^*(b,s), b) - d(y^*(b,s), s).$$

By the Maximum Theorem, w is continuous.

Paralleling our exposition in Section 1.2, we may define benchmark, ex-ante optimal solutions as follows.

Definition 4 *An* ***ex-ante stable and feasible bargaining outcome*** *for* (μ, ν, w) *is a pair* (π^*, ψ^*)*, such that* $\pi^* \in \Pi(\mu, \nu)$ *attains*

$$\max_{\pi \in \Pi(\mu,\nu)} \int_{B \times S} w \, d\pi,$$

and the function ψ^**,* $(-w)$*-convex with respect to* $\mathrm{Supp}(\mu)$ *and* $\mathrm{Supp}(\nu)$*, attains*

$$\min_{\{\psi | \psi \, is \, (-w)-convex \; w.r.t. \, \mathrm{Supp}(\mu) \; and \, \mathrm{Supp}(\nu)\}} \left(-\int_S \psi^{-w} \, d\nu + \int_B \psi \, d\mu \right).$$

Applying Proposition 1 to (μ, ν, w) yields existence of an ex-ante stable and feasible bargaining outcome, along with a closed set $\Gamma \subset B \times S$ that contains the support of any ex-ante optimal coupling, and which is contained in the $(-w)$-subdifferential of any optimal $(-w)$-convex buyer rent function.

For future reference we mention two immediate consequences. For any ex-ante stable and feasible bargaining outcome (π^*, ψ^*), it holds

$$\begin{aligned} -(\psi^*)^{-w}(s) + \psi^*(b) &= w(b,s) \quad \text{for all } (b,s) \in \mathrm{Supp}(\pi^*) \\ -(\psi^*)^{-w}(s) + \psi^*(b) &\geq w(b,s) \quad \text{for all } b \in \mathrm{Supp}(\mu), s \in \mathrm{Supp}(\nu). \end{aligned} \tag{4}$$

The benchmark interpretation is obvious. If buyers and sellers were able to match based on types (without search frictions) and to write binding contracts about partnership and investment at the ex-ante stage, the ex-ante stable and feasible bargaining outcomes would be the (socially optimal) solutions of the corresponding assignment game.

3 Ex-Ante Optimal Solutions As Investment Equilibria

This section contains our first main result. Any ex-ante stable and feasible bargaining outcome can be supported by an investment equilibrium, provided that a very mild technical condition is satisfied. This extends a main result of (CMP), namely their Proposition 3, to our very general setting. So, the result does neither hinge on one-dimensional type and attribute spaces nor on the assumptions of supermodularity of value and cost functions. This is particularly remarkable as these strong structural assumptions were used a lot in the proof of Cole, Mailath and Postlewaite (2001).

By the Maximum Theorem the solution correspondence for the problem (3) is upper-hemicontinuous (and hence a measurable selection always exists).

Take an ex-ante stable and feasible bargaining outcome (π^*, ψ^*) and suppose that the following condition is satisfied.

Condition 1 *There is a selection (β^*, σ^*) from the solution correspondence for the problem* (3) *such that $(\beta^*, \sigma^*, \pi^*)$ is a regular investment profile.*

Theorem 1 *Let (π^*, ψ^*) be an ex-ante stable and feasible bargaining outcome for (μ, ν, w). Assume that Condition 1 is satisfied and let (β^*, σ^*) be the corresponding selection. Then the regular investment profile $(\beta^*, \sigma^*, \pi^*)$ is part of an investment equilibrium $((\beta^*, \sigma^*, \pi^*), (\widetilde{\pi}^*, \widetilde{\psi}^*))$ with $\widetilde{\pi}^* = (\beta^*, \sigma^*)_{\#}\pi^*$.*

Proof of Theorem 1. So let (β^*, σ^*) be such that for all $(b, s) \in B \times S$

$$v(\beta^*(b,s), \sigma^*(b,s)) - c(\beta^*(b,s), b) - d(\sigma^*(b,s), s) \;=\; w(b,s),$$

and moreover $(\beta^*, \sigma^*, \pi^*)$ is a regular investment profile.

The measure $\widetilde{\pi}^* := (\beta^*, \sigma^*)_{\#}\pi^*$ couples $\beta^*_{\#}\pi^*$ and $\sigma^*_{\#}\pi^*$, and it is intuitively quite clear that this must be an optimal coupling. Indeed, from a social planner's point of view, and modulo technical details, the problem of finding an ex-ante optimal coupling with corresponding mutually optimal investments is equivalent to a two-stage optimization problem where he must first decide on investments for all agents and then match the two resulting populations optimally.

We now begin with the construction of a $(-v)$-convex buyer rent function $\widetilde{\psi}^*$ that will be the other part of a stable and feasible bargaining outcome that supports the investment equilibrium. That $\widetilde{\pi}^*$ is indeed optimal will be proven along the way. It should be kept in mind that in general there may be optimal couplings other than $\widetilde{\pi}^*$ for $(\beta^*_{\#}\pi^*, \sigma^*_{\#}\pi^*, v)$. Moreover, for any optimal allocation there may still be many different buyer rents that turn it into a stable and feasible bargaining outcome.

For any x for which there is some $(b,s) \in \mathrm{Supp}(\pi^*)$ such that $x = \beta^*(b,s)$, we set

$$\widetilde{\psi}^*(x) := \psi^*(b) + c(x,b).$$

This is well-defined. Indeed, assume there is some other $(b',s') \in \mathrm{Supp}(\pi^*)$ with $x = \beta^*(b',s')$. Since ψ^* is a $(-w)$-convex dual solution, we have $\mathrm{Supp}(\pi^*) \subset \partial_{-w}\psi^*$. Thus $v(x,\sigma^*(b,s)) - c(x,b) - d(\sigma^*(b,s),s) = w(b,s) = -(\psi^*)^{-w}(s) + \psi^*(b)$. Moreover, $v(x,\sigma^*(b,s)) - c(x,b') - d(\sigma^*(b,s),s) \leq w(b',s) \leq -(\psi^*)^{-w}(s) + \psi^*(b')$, where the first inequality follows from the definition of w and the second follows from (4). This implies $c(x,b) - c(x,b') \leq \psi^*(b') - \psi^*(b)$, and hence $\psi^*(b) + c(x,b) \leq \psi^*(b') + c(x,b')$. Reversing roles in the above argument shows that $\widetilde{\psi}^*(x)$ is well-defined.

Similarly, for any y for which there is some $(b,s) \in \mathrm{Supp}(\pi^*)$ such that $y = \sigma^*(b,s)$,

$$\widetilde{\phi}^*(y) := -(\psi^*)^{-w}(s) + d(y,s)$$

is well-defined.

$\widetilde{\psi}^*(x)$ and $\widetilde{\phi}^*(y)$ are the gross rents that agents get in their ex-ante efficient matches if the net rents are ψ^* and $-(\psi^*)^{-w}$ respectively.

From the equality in (4) and from the definitions of $\widetilde{\psi}^*$ and $\widetilde{\phi}^*$ it follows that for all $(b,s) \in \mathrm{Supp}(\pi^*)$,

$$\begin{aligned} v(\beta^*(b,s),\sigma^*(b,s)) &= w(b,s) + c(\beta^*(b,s),b) + d(\sigma^*(b,s),s) \\ &= \psi^*(b) - (\psi^*)^{-w}(s) + c(\beta^*(b,s),b) + d(\sigma^*(b,s),s) \\ &= \widetilde{\psi}^*(\beta^*(b,s)) + \widetilde{\phi}^*(\sigma^*(b,s)). \end{aligned} \tag{5}$$

Moreover, by the inequality in (4), we have for any $x = \beta^*(b,s)$ and $y = \sigma^*(b',s')$ with $(b,s),(b',s') \in \mathrm{Supp}(\pi^*)$,

$$\begin{aligned} \widetilde{\psi}^*(x) + \widetilde{\phi}^*(y) &= \psi^*(b) + c(x,b) - (\psi^*)^{-w}(s') + d(y,s') \\ &\geq w(b,s') + c(x,b) + d(y,s') \geq v(x,y). \end{aligned} \tag{6}$$

(5) and (6) imply that $\widetilde{\psi}^*$ is a $(-v)$-convex function, and $-\widetilde{\phi}^*$ is its $(-v)$-transform with respect to the sets $\beta^*(\mathrm{Supp}(\pi^*))$ and $\sigma^*(\mathrm{Supp}(\pi^*))$. Furthermore, by (5), the set $(\beta^*,\sigma^*)(\mathrm{Supp}(\pi^*))$, which by Lemma 2 is dense in $\mathrm{Supp}(\widetilde{\pi}^*)$, is contained in the $(-v)$-subdifferential of $\widetilde{\psi}^*$. Extending $\widetilde{\psi}^*$ as in Lemma 3 then yields the stable and feasible bargaining outcome $(\widetilde{\pi}^*, \widetilde{\psi}^*)$.

It remains to be shown that no agent has an incentive to deviate. So assume that there is a buyer of type $b \in \mathrm{Supp}(\mu)$ for whom it is profitable to deviate. Then there must be some $x \in X$ such that

$$\sup_{y \in \mathrm{Supp}(\sigma^*_{\#}\pi^*)} \Big(v(x,y) + (\widetilde{\psi}^*)^{-v}(y) \Big) - c(x,b) > \psi^*(b),$$

and hence

$$v(x,y) + (\widetilde{\psi}^*)^{-v}(y) - c(x,b) > \psi^*(b) \text{ for some } y \in \mathrm{Supp}(\sigma^*_{\#}\pi^*).$$

Since $\sigma^*(\mathrm{Supp}(\pi^*))$ is dense in $\mathrm{Supp}(\sigma^*_{\#}\pi^*)$ and by continuity of v and $(\widetilde{\psi}^*)^{-v}$, it follows that there is some $(b',s') \in \mathrm{Supp}(\pi^*)$ such that

$$v(x, \sigma^*(b',s')) + (\psi^*)^{-w}(s') - d(\sigma^*(b',s'),s') - c(x,b) > \psi^*(b).$$

and hence in particular

$$w(b,s') > -(\psi^*)^{-w}(s') + \psi^*(b),$$

which contradicts (4). ■

4 Inefficiencies And Constrained Efficiency Properties Of Investment Equilibria

We have seen that it is always possible to sustain an ex-ante efficient matching and jointly efficient investments by an investment equilibrium (Theorem 1). This is only one possible scenario though and we should raise the question whether inefficiencies can arise in equilibrium, and if so to what extent.

Conceptually, it seems useful to distinguish two different sources of inefficiency. First, agents who match in equilibrium might do so with attributes that are not jointly optimal for their relationship, i.e. with attributes that do not maximize (3). We call this **inefficiency of joint investments**. Second, the matching based on attributes might not be compatible with any ex-ante optimal matching. Formally,

Definition 5 *The investment equilibrium* $((\beta, \sigma, \pi), (\widetilde{\pi}, \widetilde{\psi}))$ *is* ***match-compatible*** *with the ex-ante coupling* $\pi' \in \Pi(\mu, \nu)$ *if for all* $(b, s), (b', s') \in \mathrm{Supp}(\pi)$ *and all* $(x, y) = (\beta(b, s), \sigma(b', s')) \in \mathrm{Supp}(\widetilde{\pi})$ *it holds that* $(b, s') \in \mathrm{Supp}(\pi')$*.*

Definition 6 *The investment equilibrium* $((\beta, \sigma, \pi), (\widetilde{\pi}, \widetilde{\psi}))$ *displays* ***mismatch inefficiency*** *if it is not match-compatible with any ex-ante optimal coupling.*

It is known from (CMP) that inefficiency of joint investments may sometimes occur. In fact, they give two examples of such equilibria, one in which parts of both populations overinvest relative to efficient levels and one in which parts of both populations underinvest. General results are hard to get and rare however. The next lemma rephrases the main abstract result of (CMP) in our framework. It is a rather indirect "constrained efficiency" property of investment equilibria. Roughly speaking, it expresses that inefficiencies are possible only if the equilibrium attribute market lacks attributes that would otherwise trigger more efficient investments (by the other side of the market): if there is a pair of agents that would ex-ante block the investment equilibrium outcome, then none of the attributes they could use for blocking may be present in the equilibrium attribute market. For completeness, we provide a short proof in the Appendix.

Lemma 4 (Lemma 2 of (CMP)) *Let* $((\beta, \sigma, \pi), (\widetilde{\pi}, \widetilde{\psi}))$ *be an investment equilibrium. Suppose that there are* $b \in \mathrm{Supp}(\mu)$*,* $s \in \mathrm{Supp}(\nu)$ *and* $(x, y) \in X \times Y$ *such that* $v(x, y) - c(x, b) - d(y, s) > r_B(b) + r_S(s)$*. So,* (b, s) *would ex-ante block the investment equilibrium and share the additional net value in some way. Then,* $x \notin \mathrm{Supp}(\beta_{\#}\pi)$ *and* $y \notin \mathrm{Supp}(\sigma_{\#}\pi)$*.*

Proof. See Appendix. ■

Mismatch inefficiency is impossible in (CMP). In their setting, investment functions are strictly increasing. Then, as v is strictly supermodular the resulting match is always positively assortative based on attributes and hence also based on types. Moreover, since costs are strictly submodular in attribute choice and type, this is match-compatible with the ex-ante efficient coupling. We show in Section 5 below that for a 1-d model that generalizes the one of (CMP) (with strictly supermodular value and costs satisfying the Milgrom-Shannon (1994) single-crossing condition, dropping convexity and smoothness assumptions) every investment equilibrium is match-compatible with the ex-ante optimal coupling, which is the positively assortative one. This holds true for arbitrary measures μ and ν, in particular measure supports need not be connected and there may be atoms. So, mismatch inefficiency can arise only in cases beyond the "1-d supermodular model".

Cole, Mailath and Postlewaite use Lemma 4 to argue that the inefficiencies in their examples disappear when the agent populations become very heterogenous, since this necessarily leads to "sufficiently diverse" attribute markets[5].

We next provide a simple but useful sufficient condition on the match value v, as well as the costs c and d that rules out inefficiency of joint investments. The starting point is the observation that in any investment equilibrium, given the attribute of their equilibrium match, agents fully internalize the net social consequences of their attribute choice.

Lemma 5 *In an investment equilibrium* $((\beta, \sigma, \pi), (\widetilde{\pi}, \widetilde{\psi}))$, *any* $(\beta(b, s), y) \in \mathrm{Supp}(\widetilde{\pi})$, *where* $(b, s) \in \mathrm{Supp}(\pi)$, *satisfies* $\beta(b, s) \in \mathrm{argmax}_{x \in X} (v(x, y) - c(x, b))$. *Similarly, for any* $(b', s') \in \mathrm{Supp}(\pi)$ *and any* $(x, \sigma(b', s')) \in \mathrm{Supp}(\widetilde{\pi})$, *it holds that* $\sigma(b', s') \in \mathrm{argmax}_{y \in Y} (v(x, y) - d(y, s'))$.

Proof. See Appendix. ■

Definition 7 *The pair of attributes* (x, y) *is* ***one-sided stable*** *for* $(b, s) \in B \times S$ *if it satisfies both* $x \in \mathrm{argmax}_{\hat{x} \in X} (v(\hat{x}, y) - c(\hat{x}, b))$ *and* $y \in \mathrm{argmax}_{\hat{y} \in Y} (v(x, \hat{y}) - d(\hat{y}, s))$.

[5](CMP) employ a corollary of Lemma 4, namely their Proposition 4, which is tailored to the case of no mismatch inefficiency.

Corollary 1 *Assume that for all $b \in \mathrm{Supp}(\mu)$, $s \in \mathrm{Supp}(\nu)$, there is a unique one-sided stable pair of attributes (which then coincides with the unique pair $(x^*(b,s), y^*(b,s))$ of jointly optimal investments). Then there is no investment equilibrium that exhibits inefficiency of joint investments.*

Proof. Immediate from Lemma 5. ■

The condition that there be a unique one-sided stable pair of attributes for (b,s) means that the complete information game between players b and s with strategy spaces X and Y and payoffs $v(x,y) - c(x,b)$ and $v(x,y) - d(y,s)$ has a unique Nash equilibrium. In particular, when v is supermodular (expressing complementarity of investments), all these games are supermodular games for which multiplicity of equilibria is known to be common rather than exceptional. Still, equilibrium uniqueness for all (b,s) is not so unusual for some "classical" combinations of cost and value functions used in economic models, as is shown by the first example in (CMP), as well as by our computations in Example 1 below.

5 The 1-d Model With Strictly Supermodular Value And Single-Crossing Costs

In this section, the type and attribute spaces are one-dimensional, $X, Y, B, S \subset \mathbb{R}_+$. We maintain the general assumptions about v, c and d. In addition, we impose some supermodularity and single-crossing conditions. This yields a model that generalizes the one used in (CMP) in three respects: no smoothness is assumed, costs need not be convex in attribute choice and must only satisfy a single-crossing condition, and types need not be uniformly distributed on intervals.

Assumption 1 *v is strictly supermodular in (x,y), $(-c)$ satisfies the Milgrom-Shannon strict single crossing condition in (x,b), and $(-d)$ satisfies the Milgrom-Shannon strict single crossing condition in (y,s)* [6].

[6] For formal definitions of these well-known concepts see for instance Milgrom and Shannon (1994) or Topkis (1998) (for the single-crossing condition), and Milgrom and Roberts (1990) or Topkis (1998) (for supermodularity).

We begin with a result that is well known: strict supermodularity of v forces optimal couplings to be positively assortative for any attribute economy. The Kantorovich duality theorem can be used for a very short proof.

Lemma 6 *For any attribute economy $(\widetilde{\mu}, \widetilde{\nu}, v)$, the optimal coupling is the positively assortative one.*

Proof of Lemma 6. By Kantorovich duality, the support of any optimal coupling $\widetilde{\pi}$ is a $(-v)$-cyclically monotone set. In particular, for any $(x, y), (x', y') \in \mathrm{Supp}(\widetilde{\pi})$ with $x > x'$, it holds $v(x, y) + v(x', y') \geq v(x, y') + v(x', y)$ and hence $v(x, y) - v(x', y) \geq v(x, y') - v(x', y')$. Since v has strictly increasing differences, it follows that $y \geq y'$. ■

Lemma 7 *Attribute choices are monotone non-decreasing in type in any investment equilibrium.*

Proof of Lemma 7. From Definition 3, $\beta(b, s) \in \mathrm{argmax}_{x \in X}(\widetilde{r}_B(x) - c(x, b))$. The objective satisfies the Milgrom-Shannon strict single-crossing property in (x, b). By Theorem 2.8.7 from Topkis (1998), all selections from the solution correspondence are monotone non-decreasing in b. The argument for sellers is analogous. ■

Corollary 2 *Every investment equilibrium $((\beta, \sigma, \pi), (\widetilde{\pi}, \widetilde{\psi}))$ is match-compatible with the positively assortative coupling of the ex-ante populations.*

Proof of Corollary 2. Follows directly from Lemma 6 and Lemma 7. ■

Note that the positively assortative coupling may of course feature matching buyers of the same type to different seller types, and vice versa, whenever the distributions have atoms, but this does not matter for the result.

Proposition 2 *Assume that the assumptions of Corollary 1 are satisfied. Then every investment equilibrium is ex-ante efficient.*

Proof of Proposition 2. By Corollary 2 buyer and seller types can ex-post be interpreted to be coupled in the same way (positively assortative) in any investment equilibrium. In particular, they are coupled in the same way as in the ex-ante efficient investment equilibrium that we constructed in Section 3 (note that, by (upper hemi-) continuity, Condition 1 is automatically satisfied if $(x^*(b,s), y^*(b,s))$ is unique for all (b,s)). By Corollary 1, all agents make jointly optimal investments for their equilibrium partnership in every investment equilibrium. This proves the claim. ■

Assume that Supp(μ) and Supp(ν) are intervals (apart from the isolated void types of course). In the underinvestment equilibrium of (CMP), pairs of low type agents invest efficiently, but higher types underinvest because they fail to coordinate on an upward jump to a more profitable one-sided stable pair of attributes. The significantly higher, and hence far more costly, investments only pay off if both sides invest accordingly so that the gains from a regime of increased complementarity in the match value function can be realized. However, this type of inefficiency becomes impossible whenever the agent populations are sufficiently heterogenous. In the presence of sufficiently high type agents, an upward jump in attributes **must** occur somewhere in the population, but if it does, it must happen for the then "mediocre" lowest types who previously underinvested: the inefficiency unravels. Analogously, for populations consisting of high and mediocre agents, (CMP) identify an equilibrium in which high agents invest efficiently and mediocre agents overinvest. It unravels when populations are augmented to include sufficiently low types.

One might conjecture that it is generally true that there can be only underinvestment or only overinvestment and that sufficient heterogeneity always rules out inefficiencies. However, both of these conjectures are false. Even in the 1-d supermodular setting of this section, the scope of inefficiency in equilibrium depends in a complex manner on both technology (which determines multiplicity of one-sided stable pairs of attributes, etc.) and population measures. Staying close to the examples of (CMP), we demonstrate a very interesting type of possible inefficiency by means of Example 1. It may happen that "lower mediocrity" underinvests and hence in some sense bunches with lower types who invest efficiently, while "upper mediocrity" overinvests and bunches with high types who invest efficiently. Bunching is not to be taken literally here since attributes are still fully differentiated, it is rather to be understood as bunching in the same connected component of the attribute economy. Hence, **the attribute economy fails to have an efficient middle sector**. Moreover, this type of inefficient equilibrium does not necessarily disappear when populations are altered to become more heterogenous. For the example that follows, we adopt the approach of (CMP) of piecewise construction of a supermodular value function with different regimes of complementarity. As (CMP) noted, this is only for analytical convenience. One could easily smooth out the non-differentiable parts without affecting results.

Example 1 *Let* $0 < \alpha_1 < \alpha_2 < \alpha_3 < 2$, $\gamma_1, \gamma_2, \gamma_3 > 0$ *and* $f_1(z) = \gamma_1 z^{\alpha_1}$, $f_2(z) = \gamma_2 z^{\alpha_2}$, $f_3(z) = \gamma_3 z^{\alpha_3}$. *For* $i < j$ *there is a unique* $z_{ij} \in \mathbb{R}_{++}$ *in which* f_j *crosses* f_i *(from below).* z_{ij} *is given by*

$$z_{ij} = \left(\frac{\gamma_i}{\gamma_j}\right)^{\frac{1}{\alpha_j - \alpha_i}}.$$

We are interested in parameter constellations for which $z_{12} < z_{23}$, *so that* f_2 *crosses* f_1 *first, then* f_3 *crosses* f_1 *and finally* f_3 *crosses* f_2. *In this case,*

$$v(x, y) := (\max(f_1, f_2, f_3))\,(xy)$$

defines a strictly supermodular value function in $(x, y) \in \mathbb{R}^2_{++}$ *(for the present example, there is no need to specify compact* X, Y, *we just let* $X = Y = \mathbb{R}_{++}$*). Indeed,* $f_1(xy)$ *is strictly increasing and strictly supermodular in* (x, y), *and* $v(x, y)$ *satisfies* $v(x, y) = g(f_1(xy))$ *for the strictly increasing, convex function*

$$g(t) = \begin{cases} t & \text{for } t \le \gamma_1 z_{12}^{\alpha_1} \\ \gamma_1^{-\alpha_2/\alpha_1} \gamma_2 t^{\alpha_2/\alpha_1} & \text{for } \gamma_1 z_{12}^{\alpha_1} < t \le \gamma_1 z_{23}^{\alpha_1} \\ \gamma_1^{-\alpha_3/\alpha_1} \gamma_3 t^{\alpha_3/\alpha_1} & \text{for } t > \gamma_1 z_{23}^{\alpha_1}. \end{cases}$$

The claim thus follows, e.g from an adaptation of Lemma 2.6.4 in Topkis (1998).

We assume that costs are given by $c(x, b) = x^4/b^2$ *and* $d(y, s) = y^4/s^2$ *for* $b, s \in \mathbb{R}_{++}$. *Population measures are symmetric,* $\mu = \nu$, *with interval support* $I \subset \mathbb{R}_{++}$. *By Corollary 2,* b *is matched to* $s = b$ *in any investment equilibrium.*

If the value function is globally given by $f_i(xy)$ *rather than* v, *then there are unique one-sided stable attributes for all* (b, s). *Indeed, consider the problem*

$$\max_{x,y \in \mathbb{R}_{++}} \left(\gamma_i (xy)^{\alpha_i} - \frac{x^4}{b^2} - \frac{y^4}{s^2}\right).$$

By behavior of the objective function on the main diagonal $x = y$ for small x, as well as by the asymptotic behavior as $x \to \infty$ or $y \to \infty$, there is an interior global maximum. Necessary first order conditions are

$$\begin{cases} \gamma_i \alpha_i x^{\alpha_i - 1} y^{\alpha_i} &= \frac{4}{b^2} x^3 \\ \gamma_i \alpha_i x^{\alpha_i} y^{\alpha_i - 1} &= \frac{4}{s^2} y^3 \end{cases} \Rightarrow \begin{cases} y &= (\frac{4}{\gamma_i \alpha_i b^2})^{1/\alpha_i} x^{(4-\alpha_i)/\alpha_i} \\ x &= (\frac{4}{\gamma_i \alpha_i s^2})^{1/\alpha_i} y^{(4-\alpha_i)/\alpha_i}. \end{cases}$$

Plugging in yields a unique stationary point (x_i, y_i) satisfying

$$\begin{cases} x_i^{(4-\alpha_i)^2/\alpha_i^2 - 1} &= (\frac{\gamma_i \alpha_i s^2}{4})^{1/\alpha_i} (\frac{\gamma_i \alpha_i b^2}{4})^{(4-\alpha_i)/\alpha_i^2} \\ y_i^{(4-\alpha_i)^2/\alpha_i^2 - 1} &= (\frac{\gamma_i \alpha_i b^2}{4})^{1/\alpha_i} (\frac{\gamma_i \alpha_i s^2}{4})^{(4-\alpha_i)/\alpha_i^2}. \end{cases}$$

In particular, for $s = b$, we obtain

$$x_i = y_i = \left(\frac{\gamma_i \alpha_i b^2}{4} \right)^{\frac{4/\alpha_i^2}{(4-\alpha_i)^2/\alpha_i^2 - 1}} = \left(\frac{\gamma_i \alpha_i b^2}{4} \right)^{\frac{1}{4 - 2\alpha_i}}. \tag{7}$$

The net value that the pair (b, b) generates with these maximizing attributes (for value function $f_i(xy)$!) is

$$\gamma_i x_i^{2\alpha_i} - 2\frac{x_i^4}{b^2} = \gamma_i \left(\frac{\gamma_i \alpha_i b^2}{4} \right)^{\frac{\alpha_i}{2-\alpha_i}} - \frac{2}{b^2} \left(\frac{\gamma_i \alpha_i b^2}{4} \right)^{\frac{2}{2-\alpha_i}} = \kappa_i b^{\frac{2\alpha_i}{2-\alpha_i}} =: w_i(b),$$

where

$$\kappa_i = \gamma_i^{\frac{2}{2-\alpha_i}} \left(\left(\frac{\alpha_i}{4} \right)^{\frac{\alpha_i}{2-\alpha_i}} - 2 \left(\frac{\alpha_i}{4} \right)^{\frac{2}{2-\alpha_i}} \right) = \gamma_i^{\frac{2}{2-\alpha_i}} \left(\frac{\alpha_i}{4} \right)^{\frac{\alpha_i}{2-\alpha_i}} \left(1 - \frac{\alpha_i}{2} \right).$$

Observe that for $i < j$, w_j crosses w_i exactly once in $\mathbb{R}_{++}$ and this crossing is from below (like for the functions f_i, f_j). The critical types are given by

$$b_{ij} = \left(\frac{\kappa_i}{\kappa_j} \right)^{\frac{1}{2\alpha_j/(2-\alpha_j) - 2\alpha_i/(2-\alpha_i)}} = \left(\frac{\kappa_i}{\kappa_j} \right)^{\frac{(2-\alpha_i)(2-\alpha_j)}{4(\alpha_j - \alpha_i)}}.$$

So, if the agent pair (b, b) were free to choose among the (globally valid) gross values f_i and f_j, then for $b < b_{ij}$ they would choose f_i, and for $b > b_{ij}$ they would choose f_j. We may then compare the attributes that the indifferent types b_{ij} would use under f_i, *denoted as x_{iij} to those they would use under f_j, denoted as x_{jij}.*

$$
\begin{aligned}
x_{iij} &= \left(\frac{\gamma_i \alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\kappa_i}{\kappa_j}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}} \\
&= \gamma_i^{\frac{1}{4-2\alpha_i}+\frac{2-\alpha_j}{2(2-\alpha_i)(\alpha_j-\alpha_i)}} \gamma_j^{-\frac{1}{2(\alpha_j-\alpha_i)}} \left(\frac{\alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}} \\
&= \left(\frac{\gamma_i}{\gamma_j}\right)^{\frac{1}{2(\alpha_j-\alpha_i)}} \left(\frac{\alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}} \\
&= z_{ij}^{\frac{1}{2}} \left(\frac{\alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}}.
\end{aligned}
$$

A similar computation yields

$$
\begin{aligned}
x_{jij} &= \left(\frac{\gamma_j \alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\kappa_i}{\kappa_j}\right)^{\frac{2-\alpha_i}{4(\alpha_j-\alpha_i)}} \\
&= \gamma_i^{\frac{1}{2(\alpha_j-\alpha_i)}} \gamma_j^{\frac{1}{4-2\alpha_j}-\frac{2-\alpha_i}{2(2-\alpha_j)(\alpha_j-\alpha_i)}} \left(\frac{\alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_i}{4(\alpha_j-\alpha_i)}} \\
&= z_{ij}^{\frac{1}{2}} \left(\frac{\alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_i}{4(\alpha_j-\alpha_i)}}.
\end{aligned}
$$

Thus, x_{iij} and x_{jij} depend on γ_i, γ_j only through γ_i/γ_j, and moreover, x_{jij}/x_{iij} depends only on α_i and α_j. We get

$$
\frac{x_{jij}}{x_{iij}} = \left(\frac{\alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\alpha_i}{4}\right)^{-\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{1}{4}} = \left(\frac{\alpha_j(2-\alpha_i)}{\alpha_i(2-\alpha_j)}\right)^{\frac{1}{4}}.
$$

This ratio is greater than one (since $0 < \alpha_i < \alpha_j < 2$), so that there is an upward jump in attribute choice where the indifferent pair would switch from the f_i to the f_j value function.

We are interested in parameter constellations for which $b_{12} < b_{23}$, so that for $b < b_{12}$, f_1 would be the best value function, for $b_{12} < b < b_{23}$, f_2 would be best, and for $b_{23} < b$, f_3 would be best.

However, the true gross value function is v with its three different regimes of complementarity! Assume that we want to find the ex-ante efficient investment equilibrium *first. To this end, the comparison of net values from unconstrained optimal choices for f_1, f_2 and f_3 is justified only if the "jump attributes" actually lie in the valid regimes. Formally, we need*

$$x_{112}^2 < z_{12} < x_{212}^2 < x_{223}^2 < z_{23} < x_{323}^2. \tag{8}$$

For the case we consider, i.e. $b_{12} < b_{13} < b_{23}$, it is clear from (7) that $x_{112} < x_{113}$, $x_{212} < x_{223}$ and $x_{313} < x_{323}$. We show now that the following two conditions may simultaneously be satisfied

a) (8) holds

b) the jump from x_{113} to x_{313} (which is not part of the efficient equilibrium!) is also between valid regimes, that is

$$x_{113}^2 < z_{12} \text{ and } z_{23} < x_{313}^2.$$

Indeed, as an example, let $\alpha_1 = 0.1$, $\alpha_2 = 0.6$, $\alpha_3 = 1.6$, $\gamma_1 = 1$, $\gamma_2 = 1.5$ and $\gamma_3 = 1$. Then $z_{12} = 4/9$, $z_{23} = 3/2$, $b_{12} \approx 1.5823$, $b_{13} \approx 1.8908$, $b_{23} \approx 1.9266$ and jump attributes for all three possible jumps lie in the valid regimes: $x_{112}^2 \approx 0.2326$, $x_{113}^2 \approx 0.2806$, $x_{212}^2 \approx 0.6637$, $x_{223}^2 \approx 0.8793$, $x_{313}^2 \approx 2.4459$ and $x_{323}^2 \approx 2.6863$.

To complete the example, it remains to be shown that for the above parameters and any symmetric populations with interval support I and $b_{13} \in I$, the inefficient outcome in which types $b < b_{13}$ make investments $\beta(b) = \sigma(b) = \left(\frac{\gamma_1\alpha_1 b^2}{4}\right)^{\frac{1}{4-2\alpha_1}}$ and types $b > b_{13}$ make investments $\beta(b) = \sigma(b) = \left(\frac{\gamma_3\alpha_3 b^2}{4}\right)^{\frac{1}{4-2\alpha_3}}$ can be sustained as an investment equilibrium. We provide a complete proof in the Appendix, for the symmetric rents $\widetilde{\psi}(x) = \widetilde{\phi}(x) = v(x,x)/2$ on $cl(\beta(I)) = cl(\sigma(I))$ ($cl(\cdot)$ denotes the closure of a set).

6 Appendix

Proof of Lemma 1. We prove the claim for μ only and first show $\mathrm{P}_B(\mathrm{Supp}(\pi)) \subset \mathrm{Supp}(\mu)$. So take any $(b,s) \in \mathrm{Supp}(\pi)$. Then, for any open neighborhood U of b, $\pi(U \times S) > 0$ and hence $\mu(U) > 0$. Thus, $b \in \mathrm{Supp}(\mu)$.

We next prove the slightly less trivial inclusion $\mathrm{Supp}(\mu) \subset \mathrm{P}_B(\mathrm{Supp}(\pi))$.

Assume to the contrary that there is some $b \in \mathrm{Supp}(\mu)$ that is not contained in $\mathrm{P}_B(\mathrm{Supp}(\pi))$. The latter assumption implies that for all $s \in S$ there are open neighborhoods $U_s \subset B$ of b and $V_s \subset S$ of s such that $\pi(U_s \times V_s) = 0$. Since S is compact, the open cover $\{V_s\}_{s\in S}$ of S contains a finite subcover $\{V_{s_1}, ..., V_{s_k}\}$. Moreover, $U := \bigcap_{i=1}^k U_{s_i}$ is an open neighborhood of b and $U \times S \subset \bigcup_{i=1}^k U_{s_i} \times V_{s_i}$. Hence we get the contradiction $0 < \mu(U) = \pi(U \times S) \leq \pi(\bigcup_{i=1}^k U_{s_i} \times V_{s_i}) = 0$. ■

Proof of Lemma 2. We prove the claim for $\beta(\text{Supp}(\pi))$. Assume to the contrary that there is a point $x \in \text{Supp}(\beta_\# \pi)$ and an open neighborhood U of x such that $U \cap \beta(\text{Supp}(\pi)) = \emptyset$. Then $\beta_\#\pi(U) > 0$ (by definition of the support) and on the other hand $\beta_\#\pi(U^c) \geq \pi(\text{Supp}(\pi)) = 1$. Contradiction. ■

Proof of Lemma 3. We first define for all $y \in \text{Supp}(\sigma_\#\pi)$,

$$\widetilde{\phi}_0(y) := \sup_{x\in\beta(\text{Supp}(\pi))} (v(x,y) - \widetilde{\psi}(x)).$$

By definition, $\widetilde{\phi}_0$ coincides with $-\widetilde{\psi}^{-v}$ on the set $\sigma(\text{Supp}(\pi)) \subset \text{Supp}(\sigma_\#\pi)$, which is a dense subset by Lemma 2. Next we set for all $x \in \text{Supp}(\beta_\#\pi)$,

$$\widetilde{\psi}_1(x) := \sup_{y\in\text{Supp}(\sigma_\#\pi)} (v(x,y) - \widetilde{\phi}_0(y)),$$

and finally for all $y \in \text{Supp}(\sigma_\#\pi)$,

$$\widetilde{\phi}_1(y) := \sup_{x\in\text{Supp}(\beta_\#\pi)} (v(x,y) - \widetilde{\psi}_1(x)).$$

By definition, $\widetilde{\psi}_1$ is a $(-v)$-convex function with respect to the compact metric spaces $\text{Supp}(\beta_\#\pi)$ and $\text{Supp}(\sigma_\#\pi)$, and $-\widetilde{\phi}_1$ is its $(-v)$-transform. We need to check that $\widetilde{\psi}_1$ coincides with $\widetilde{\psi}$ on $\beta(\text{Supp}(\pi))$, and moreover that $-\widetilde{\phi}_1$ coincides with $\widetilde{\psi}^{-v}$ on $\sigma(\text{Supp}(\pi))$.

To this end, observe that for any $x = \beta(b,s)$ with $(b,s) \in \text{Supp}(\pi)$, the set of real numbers used to define the supremum $\widetilde{\psi}(x)$ is contained in the one used to define $\widetilde{\psi}_1(x)$. Assume then for the sake of deriving a contradiction that $\widetilde{\psi}_1(\beta(b,s)) > \widetilde{\psi}(\beta(b,s))$. Then there must be some $y \in \text{Supp}(\sigma_\#\pi)$, such that $v(\beta(b,s),y) > \widetilde{\psi}(\beta(b,s)) + \widetilde{\phi}_0(y)$ and hence in particular $v(\beta(b,s),y) > \widetilde{\psi}(\beta(b,s)) + v(\beta(b,s),y) - \widetilde{\psi}(\beta(b,s))$, which yields a contradiction. A completely analogous argument shows that $\widetilde{\phi}_1(\sigma(b,s)) = -\widetilde{\psi}^{-v}(\sigma(b,s))$ for all $(b,s) \in \text{Supp}(\pi)$. So $\widetilde{\psi}(x) := \widetilde{\psi}_1(x)$ and $\widetilde{\psi}^{-v}(y) := -\widetilde{\phi}_1(y)$ are well-defined (and unique) extensions to a $(-v)$-dual pair with respect to $\mathrm{Supp}(\beta_{\#}\pi)$ and $\mathrm{Supp}(\sigma_{\#}\pi)$.

Since, for the extended $\widetilde{\psi}$, $\partial_{-v}\widetilde{\psi}$ is closed, it follows that $\mathrm{Supp}(\widetilde{\pi}) \subset \partial_{-v}\widetilde{\psi}$. Hence $(\widetilde{\pi}, \widetilde{\psi})$ is a stable and feasible bargaining outcome. ■

Proof of Lemma 4. Assume to the contrary that $x \in \mathrm{Supp}(\beta_{\#}\pi)$. Then, from the definition of r_S, and by assumption,

$$r_S(s) + \widetilde{\psi}(x) - c(x,b) \;\geq\; v(x,y) - \widetilde{\psi}(x) - d(y,s) + \widetilde{\psi}(x) - c(x,b) \;>\; r_B(b) + r_S(s).$$

Hence $\widetilde{\psi}(x) - c(x,b) > r_B(b)$, a contradiction (formally, $\widetilde{\psi}(x) = v(x,y') + \widetilde{\psi}^{-v}(y')$ for some $y' \in \mathrm{Supp}(\sigma_{\#}\pi)$ matched with x under $\widetilde{\pi}$ and this leads to a contradiction to the definition of r_B). The proof for $y \notin \mathrm{Supp}(\sigma_{\#}\pi)$ is analogous. ■

Proof of Lemma 5. Assume to the contrary that there is some x such that $v(x,y) - c(x,b) > v(\beta(b,s),y) - c(\beta(b,s),b)$. $(\beta(b,s),y) \in \mathrm{Supp}(\widetilde{\pi})$ implies $\widetilde{\psi}(\beta(b,s)) = v(\beta(b,s),y) + \widetilde{\psi}^{-v}(y)$. Hence

$$\begin{aligned}
\widetilde{\psi}(\beta(b,s)) - c(\beta(b,s),b) &= v(\beta(b,s),y) + \widetilde{\psi}^{-v}(y) - c(\beta(b,s),b) \\
&< v(x,y) + \widetilde{\psi}^{-v}(y) - c(x,b) \;\leq\; r_B(b),
\end{aligned}$$

which contradicts the assumption that $\beta(b,s)$ is an investment equilibrium choice of buyer b. The proof for sellers is of course entirely analogous. ■

Remainder of Example 1. It is straightforward to check that $\widetilde{\psi}$ is a $(-v)$-convex function with respect to the sets $cl(\beta(I))$ and $cl(\sigma(I)) = cl(\beta(I))$, that $\widetilde{\phi}$ is its transform, and that the deterministic coupling of the symmetric attribute measures given by the identity mapping is supported in $\partial_{-v}\widetilde{\psi}$. This yields a stable and feasible bargaining outcome for the attribute economy.

Next, we already know that for the rents $\widetilde{\phi}$, buyer type b_{13} is indifferent between the option (choose $x = x_{113}$, match with $y = x_{113}$) and the option (choose $x = x_{313}$, match with $y = x_{313}$). Indeed, net rents from this are $\gamma_1 x_{113}^{2\alpha_1}/2 - c(x_{113}, b_{13}) = w_1(b_{13})/2$ and $\gamma_3 x_{313}^{2\alpha_3}/2 - c(x_{313}, b_{13}) = w_3(b_{13})/2$ which are equal by definition of b_{13}. We first proceed to show that these are indeed the optimal choices for buyer type b_{13}. Note that for a given y, the conditionally optimal $x^*(y, b_{13})$ solves

$$\max_{x \in \mathbb{R}_{++}} \left(v(x,y) - \frac{v(y,y)}{2} - c(x, b_{13}) \right),$$

where

$$v(x,y) = \begin{cases} \gamma_1 (xy)^{\alpha_1} & \text{for } x \le z_{12}/y \\ \gamma_2 (xy)^{\alpha_2} & \text{for } z_{12}/y \le x \le z_{23}/y \\ \gamma_3 (xy)^{\alpha_3} & \text{for } z_{23}/y \le x. \end{cases}$$

Let $y \le x_{113}$. Then, $x^*(y, b_{13}) \le x_{113}$. Indeed,

$$\frac{\partial}{\partial x}\left(\gamma_i (xy)^{\alpha_i} - \frac{x^4}{b_{13}^2}\right) = \gamma_i \alpha_i y^{\alpha_i} x^{\alpha_i - 1} - \frac{4x^3}{b_{13}^2}$$

is strictly positive for $x < \left(\frac{\gamma_i \alpha_i y^{\alpha_i} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_i}}$ and strictly negative for $x > \left(\frac{\gamma_i \alpha_i y^{\alpha_i} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_i}}$. For $y \le x_{113}$, this zero is less than or equal to $\left(\frac{\gamma_i \alpha_i x_{113}^{\alpha_i} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_i}}$, which for $i = 1$ equals x_{113}. For $i = 2$, $\left(\frac{\gamma_2 \alpha_2 x_{113}^{\alpha_2} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_2}} = 0.8385 < z_{12}/x_{113} = 0.8391$, so that the derivative is negative on the entire second part of the domain. Similarly, $\left(\frac{\gamma_3 \alpha_3 x_{113}^{\alpha_3} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_3}} = 0.7599 < z_{23}/x_{113}$. It follows that $\max_{x \in \mathbb{R}_{++}, y \le x_{113}} \left(v(x,y) - \frac{v(y,y)}{2} - c(x, b_{13})\right)$ is attained in the domain of definition of v where it coincides with f_1, the first order condition then yields $y = x$ and thus (maximizing $\frac{\gamma_1 x^{2\alpha_1}}{2} - c(x, b_{13})$) $x = y = x_{113}$. A completely analogous reasoning applies for $y \ge x_{313}$ (we omit the details), showing that $\max_{x \in \mathbb{R}_{++}, y \ge x_{313}} \left(v(x,y) - \frac{v(y,y)}{2} - c(x, b_{13})\right)$ is attained at $x = y = x_{313}$.

Therefore buyer type b_{13} is indifferent between his two optimal choices (choose x_{113}, match with $y = x_{113}$) and (choose x_{313}, match with $y = x_{313}$). Next, the buyers' objective function, choosing conditionally optimal y, is strictly supermodular in (x, b), so that by Theorem 2.8.4 of Topkis (1998), any selection of optimal x is increasing in b. Consider $b < b_{13}$ then. Any optimal x must be less than or equal to x_{113}. On the other hand, the objective is strictly supermodular in (x, y), so that the optimal y also satisfies $y \le x_{113}$. Hence, optima lie in the domain of definition of f_1. First order conditions lead to $y = x$, and thus to maximization of $\gamma_1 x^{2\alpha_1}/2 - c(x, b)$ and hence to $x = \beta(b)$. The argument for buyer types $b > b_{13}$ is analogous. The entire argument applies also to sellers. This concludes the proof. QED ■

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