Optimal auction design with endogenously correlated buyer types

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Abstract

This paper studies optimal auction design when the seller can increase the buyers' valuations through an unobservable ex ante investment. The key insight is that the optimal mechanism may have the seller play a mixed investment strategy so as to create correlation between the otherwise (conditionally) independent valuations of buyers. Sufficient conditions on model parameters are derived so that the seller can, in fact, extract the first best surplus almost fully.

Keywords: Auction design, correlation, ex ante investment, moral hazard, adverse selection.

JEL Classification No.: C72, D42, D44, D82

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1 Introduction

In many situations a seller can affect buyers' valuations by an unobservable ex ante investment in the object at sale. For example, buyers' valuations for a house will depend on the effort spent by the construction company. The company's effort is typically not directly observable, nor can it be deduced from a buyer's valuation when the latter is partially subjective or/and influenced by random component's beyond the constructor's control. Similarly, in public procurement the contractor's cost may depend on, for example, infrastructure investments or the quality of services provided by the government who itself acts as the procurer. Or, on second hand markets, buyers' valuations are affected by the unobservable care with which initial owners have treated the item they sell. A final example is persuasive advertising.

What is the revenue maximizing selling mechanism in this setting? This is the question I address in this paper. I study a private value environment in which the seller's investment raises buyers' valuations stochastically. Conditional on the seller's investment, valuations are conditionally independent. I assume that the seller's investment is unobservable for buyers and that buyers' valuations is their private information. Thus, there is moral hazard on part of the seller and adverse selection on part of the buyers.

The purpose of the analysis is to point out that the optimal mechanism may have the seller play a mixed investment strategy. The basic reason for this result is that if the seller adopts a mixed investment strategy, then, because buyers cannot observe investment, their valuations become correlated in equilibrium. I demonstrate that under a large set of model parameters, the seller can exploit this correlation in the most extreme way and design a contract which extracts the full surplus that is generated by his investment. In fact, if his mixed investment strategy places almost full probability mass on the efficient investment level, then the seller extracts the first best surplus almost fully despite of moral hazard and adverse selection.

To see intuitively why correlation emerges when the seller randomizes, one may think of an urn model where each urn corresponds to a pure investment strategy by the seller. Buyers' valuations are drawn independently from one urn, but if the seller randomizes and buyers do not observe the realized investment, they do not know what the true urn is. Therefore, the realization of a buyer's valuation contains information about the true urn and thus about the valuation of the rival buyer.

The existence of full surplus extracting selling mechanisms when buyers' valuations are correlated is well–established (see Myerson, 1981, Cremer and McLean, 1988, McAffee and Perry, 1992). The basic idea of these constructions is to elicit buyers' beliefs by offering contingent lotteries whose payoffs depend on the choice of lotteries by rival buyers. Many lotteries exist which induce buyers to report their beliefs truthfully without leaving them information rents. I derive conditions such that at least one of those lotteries can be found which, in addition, makes the seller indifferent between his investment options so that playing a mixed investment strategy is optimal.

I begin the analysis with the simplest case with two buyers, two investment opportunities, and a binary distribution of buyer valuations. In a further step, I consider the extension to an arbitrary finite number of investment opportunities. In this case the seller not only has to be indifferent between two investment options but also has to (weakly) prefer them to all other investments. This increases the number of constraints on transfers. I shall show that for a large set of parameters, my result for the binary distribution case carries over. I shall also argue that the extension to more than two buyer valuations is less critical simply because then more transfers are available to meet the indifference condition for the seller.

Full surplus extraction results have come under criticism from a variety of angles. First, full surplus extraction critically relies on risk–neutrality (Roberts, 1991) or unlimited liability (Demougin and Garvie, 1991) of buyers, or on the absence of collusion by buyers (Laffont and Martimort, 2000). In principle, these concerns apply to a literal interpretation of my construction, too. However, even if the conditions for full surplus extraction are not met, often the correlation among buyer valuations can still be exploited to some extent.[1] In such situations, the seller might still want to create correlation by adopting a mixed strategy so that the spirit of my argument is likely to carry over. It is however difficult (or cumbersome) to explicitly construct optimal mechanisms for correlated valuations if full surplus extraction fails. To make my point most clearly, I therefore abstract from the above concerns and study an environment in which the optimal mechanism for correlated valuations is easy to obtain.

Second, a recent literature points out that full surplus extraction depends on the (rather demanding) assumption of common knowledge of the distribution of buyers' valuations and their higher order beliefs (Neeman, 2004, Neeman and Heifetz, 2006, Barelli, 2009) and, thus,

[1] See e.g. Bose and Zhao (2007) who study optimal design when the agents' beliefs violate the rank condition by Cremer and McLean (1988), or Dequiedt and Martimort (2009) who consider the case when the designer cannot commit to a grand mechanism but only to bilateral contracts with each agent.

is not "robust" (Bergemann and Morris, 2005). While some work studies optimal robust design (Chung and Ely, 2007, Bergemann and Schlag, 2008) it is an open issue how a seller can exploit correlation when he insists on robust design. Note, however, that in my setup the joint distribution of buyers' valuations emerges endogenously in equilibrium as a result of the seller's investment. Therefore, if there are no significant exogenous information sources that affect buyers' valuations and/or their beliefs, then the common knowledge assumption is simply embodied in the equilibrium concept, it is not an ad hoc assumption on players' exogenous beliefs.

A question related to mine has been raised in industrial economics by Spence (1975) who studies the incentives of a monopolist to invest in product quality. The difference is that in Spence the monopolist cannot use a revelation mechanism to price discriminate between consumers. There seems to be surprisingly little work in the mechanism design literature that considers optimal design with an ex ante investment by the designer. Instead, most work focusses on optimal design with ex ante investment by agents, such as investments in their valuation (Rogerson, 1992) or information acquisition (e.g. Bergemann and Välimäki, 2002, Cremer et al., 1998).

The paper is organized as follows. The next section presents the basic model. Section 3 derives the first best benchmark, and section 4 considers the optimal mechanism when the seller is restricted to a pure investment strategy. Section 5 contains the main argument when the seller can adopt a mixed strategy. Section 7 examines extensions of the basic model, and section 8 concludes.

2 Model

There are one seller and two risk–neutral buyers $i = 1, 2$. The seller has one unit of a good for sale. Buyer i's valuation for the good, θ_i, can take on a high value $\theta_H > 0$ or a low value $\theta_L \in (0, \theta_H)$. In what follows I indicate by $i, j \in \{1, 2\}$ a buyer's identity and by $k, \ell \in \{L, H\}$ a buyer's type. The distribution of buyers' valuations depends on a costly ex ante investment z by the seller. The seller has two investment opportunities, the "high" investment $\bar{z}$ and the "low" investment $\underline{z}$. Investing z costs $c(z)$, where $c(\bar{z}) = c > 0$ and $c(\underline{z})$ is normalized to zero. Given z, the probability with which a buyer has a high valuation is $p(z)$. Let $p(\bar{z}) = \bar{p}$ and $p(\underline{z}) = \underline{p}$ where $\bar{p} > \underline{p}$, that is, the seller's investment raises a buyer's valuation stochastically in a first order sense. Buyers' valuations are assumed to be conditionally independent, conditional on z. Moreover, I assume that buyers cannot directly observe the seller's investment choice.

The seller may randomize between investments. Let ζ be the probability with which the seller chooses $\bar{z}$. Let $p_{k\ell}(\zeta)$ be the joint probability with which the valuation profile $(\theta_1, \theta_2) = (\theta_k, \theta_\ell)$ occurs, conditional on ζ. Due to conditional independence, this joint distribution can be written as in Table 1:

	θ_H	θ_L
θ_H	$\bar{p}^2\zeta + \underline{p}^2(1-\zeta)$	$\bar{p}(1-\bar{p})\zeta + \underline{p}(1-\underline{p})(1-\zeta)$
θ_L	$\bar{p}(1-\bar{p})\zeta + \underline{p}(1-\underline{p})(1-\zeta)$	$(1-\bar{p})^2\zeta + (1-\underline{p})^2(1-\zeta)$

(1)

Table 1: Joint probability distribution $p_{k\ell}(\zeta)$ conditional on ζ.

I denote by $\mu_{k\ell}(\zeta)$ the belief of a buyer of type θ_k that the rival buyer is of type θ_ℓ, conditional on ζ.[2] By Bayes' rule,

$$\mu_{HH}(\zeta) = \frac{\bar{p}^2\zeta + \underline{p}^2(1-\zeta)}{\bar{p}\zeta + \underline{p}(1-\zeta)}, \tag{2}$$

$$\mu_{LH}(\zeta) = \frac{\bar{p}(1-\bar{p})\zeta + \underline{p}(1-\underline{p})(1-\zeta)}{(1-\bar{p})\zeta + (1-\underline{p})(1-\zeta)}. \tag{3}$$

For a buyer of type θ_k, denote the vector of beliefs about the rival buyer by $\mu_k(\zeta)$:

$$\mu_k(\zeta) = \begin{pmatrix} \mu_{kH}(\zeta) \\ \mu_{kL}(\zeta) \end{pmatrix}. \tag{4}$$

The main argument of the paper rests on the insight that for $\zeta \in (0,1)$ types are correlated from the point of view of buyers. Formally, it is easy to verify that $\mu_{HH}(\zeta) > \mu_{LH}(\zeta)$ for all $\zeta \in (0,1)$, that is, a high valuation buyer assigns a higher probability than a low valuation buyer to the event that he faces a high valuation rival buyer.

The reason for why correlation emerges is that a buyer cannot observe the investment realization. One may intuitively think of an urn model where each pure investment z corresponds to one urn. Buyers' valuations are drawn independently from one urn, but a buyer does not know from which one. Therefore, the realization of a buyer's own valuation contains information about the true urn and thus about the valuation of the rival buyer.

[2]For a pure investment strategy, I shall also write $p_{k\ell}(z)$ and $\mu_{k\ell}(z)$.

3 First best

As a benchmark, consider the situation in which the buyers' valuation is public information. In that case, the seller optimally offers the good to the buyer with the maximal valuation at a price equal to that valuation. Therefore, for each realization of types, the seller can extract the full surplus $\max\{\theta_1, \theta_2\}$, yielding a profit of

$$\pi^{FB}(z) = \sum_{k,\ell \in \{L,H\}} \max\{\theta_k, \theta_\ell\} p_{k\ell}(z) - c(z) \tag{5}$$

At the investment stage, the seller chooses z so as to maximize $\pi(z)$. Thus, the first best profit is

$$\pi^{FB} = \max\{\pi^{FB}(\bar{z}), \;\; \pi^{FB}(\underline{z})\}. \tag{6}$$

To reduce the number of case distinctions, I assume from now on that it is first best to choose the high investment $\bar{z}$ (the other case can be treated with analogous arguments):

Assumption 1: $\pi^{FB}(\bar{z}) > \pi^{FB}(\underline{z})$.

4 Seller's problem

I now examine the case in which the seller's investment is unobservable and the buyers' valuations are their private information. Therefore, the seller designs a mechanism which makes the assignment of the good and payments conditional on communication by the buyers. I consider the following timing.[3]

1. Seller proposes and commits to a mechanism.
2. Seller privately chooses an investment.
3. Buyers privately observe their valuation.
4. Buyers reject or accept the contract.

– If a buyer reject, he gets his outside option of zero.

5. If buyers accept, the mechanism is implemented.

In general, a mechanism specifies for each buyer a message set, the probability with which a buyer gets the object, and payments from buyers to the seller contingent on messages submitted

[3] A similar timing is adopted in Cremer et al. (1998). If the stages 1 and 2 are swapped, signaling issues may contaminate the analysis.

by the buyers in stage 5.[4] A mechanism induces a Bayesian game between the players which starts at date 2. In equilibrium the seller's investment is a best reply against buyers' reporting strategies, and buyer's reporting strategies are best replies against the seller's investment and the rival buyer's reporting strategy. The objective of the seller is to design a revenue maximizing mechanism subject to the constraint that an equilibrium is played in the induced game.[5]

The revelation principle implies that an equilibrium outcome of any mechanism can also be attained as an equilibrium outcome of a direct and incentive compatible mechanism. A direct mechanism asks each buyer to announce his type after stage 3 and before stage 4, and consists of an assignment rule

$$x_{k\ell} = (x^1_{k\ell}, x^2_{k\ell}), \qquad 0 \le x^1_{k\ell}, x^2_{k\ell} \le 1, \qquad x^1_{k\ell} + x^2_{k\ell} \le 1, \tag{7}$$

which specifies for each buyer i the probability $x^i_{k\ell}$ with which he obtains the good, conditional on the buyers' type announcements $(\theta_1, \theta_2) = (\theta_k, \theta_\ell)$. Moreover, it consists of a transfer rule

$$t_{k\ell} = (t^1_{k\ell}, t^2_{k\ell}), \tag{8}$$

which specifies for each buyer i the transfers $t^i_{k\ell}$ which he pays to the seller, conditional on the buyers' type announcements $(\theta_1, \theta_2) = (\theta_k, \theta_\ell)$. In vector notation:

$$x^1_k = \begin{pmatrix} x^1_{kH} \\ x^1_{kL} \end{pmatrix}, \quad x^2_k = \begin{pmatrix} x^2_{Hk} \\ x^2_{Lk} \end{pmatrix}, \quad t^1_k = \begin{pmatrix} t^1_{kH} \\ t^1_{kL} \end{pmatrix}, \quad t^2_k = \begin{pmatrix} t^2_{Hk} \\ t^2_{Lk} \end{pmatrix}. \tag{9}$$

A mechanism is incentive compatible if each buyer has an incentive to announce his type truthfully, given his beliefs about the rival buyer's type. Note that since a buyer's beliefs about the rival buyer's type depend upon his conjectures about the seller's investment strategy, incentive compatibility has to be defined for given conjectures.[6] I denote by $\zeta^c \in [0,1]$ the conjecture of a buyer that the seller has chosen investment $\bar{z}$ with probability ζ^c. The expected probability of winning and the expected transfers of type θ_k of buyer i, conditional on ζ^c, when

[4] In theory, a mechanism may also condition on a report by the seller since after the seller has chosen his investment, he has private information, too. Since I focus on conditions under which the seller can extract the full surplus, such a mechanism cannot improve on a mechanism which depends only on buyers' reports.

[5] Implicit in this formulation of the seller's problem is the (standard) assumption that the seller can select his most preferred equilibrium.

[6] I use the word "conjecture" for a buyer's beliefs about the seller's strategy so as to distinguish these beliefs from his beliefs about the rival buyer's type.

he announces type θ_ℓ are respectively given as

$$\sum_{m\in\{L,H\}} x^i_{\ell m}\mu_{km}(\zeta^c) = x^i_\ell \cdot \mu_k(\zeta^c), \qquad \sum_{m\in\{L,H\}} t^i_{\ell m}\mu_{km}(\zeta^c) = t^i_\ell \cdot \mu_k(\zeta^c). \tag{10}$$

The mechanism is incentive compatible, conditional on ζ^c if for all i, k, ℓ:

$$\theta_k x^i_k \cdot \mu_k(\zeta^c) - t^i_k \cdot \mu_k(\zeta^c) \geq \theta_k x^i_\ell \cdot \mu_k(\zeta^c) - t^i_\ell \cdot \mu_k(\zeta^c). \tag{11}$$

Finally, the seller needs to make sure that buyers participate in the mechanism at stage 4. The mechanism is individually rational, conditional on ζ^c, if for all i, k:[7]

$$\theta_k x^i_k \cdot \mu_k(\zeta^c) - t^i_k \cdot \mu_k(\zeta^c) \geq 0. \tag{12}$$

A mechanism that is incentive compatible and individually rational, conditional on ζ^c, is called feasible, conditional on ζ^c.

The seller's problem is to choose a mechanism (x, t) and an investment strategy $\zeta \in [0, 1]$ which maximizes his profit subject to the constraint that the mechanism is feasible, conditional on the buyers' conjecture ζ^c and the equilibrium requirement that the buyers' conjecture be correct: $\zeta^c = \zeta$. Let

$$\pi(z) \;\; = \;\; \sum_{k,\ell}[t^1_{k\ell} + t^2_{k\ell}]p_{k\ell}(z) - c(z) \tag{13}$$

be the seller's profit when he plays the pure strategy z. The seller's problem is thus

$$\max_{x,t,\zeta} \pi(\bar{z})\zeta + \pi(\underline{z})(1-\zeta) \quad s.t. \quad (11), (12), \zeta^c = \zeta. \tag{14}$$

The constraint $\zeta^c = \zeta$ can be eliminated by inserting it directly in the feasibility constraints (11), (12). In what follows, I can therefore ignore the distinction between the actual and the conjectured investment strategy and consider the problem

$$\max_{x,t,\zeta} \pi(\bar{z})\zeta + \pi(\underline{z})(1-\zeta) \qquad s.t. \tag{15}$$

$$\theta_k x^i_k \cdot \mu_k(\zeta) - t^i_k \cdot \mu_k(\zeta) \geq \theta_k x^i_\ell \cdot \mu_k(\zeta) - t^i_\ell \cdot \mu_k(\zeta) \tag{16}$$

$$\theta_k x^i_k \cdot \mu_k(\zeta) - t^i_k \cdot \mu_k(\zeta) \geq 0. \tag{17}$$

[7] If it is optimal for the seller to exclude one buyer i, he can do so by setting $x_i = t_i = 0$.

5 Mechanisms with pure investment strategies

In this section, I study the seller's problem when he is restricted to choose a pure investment strategy. I proceed by determining the optimal mechanism for a given z, and then compare the seller's profit for low and high investment. A crucial observation is that if the seller selects a pure strategy, then due to conditional independence a buyer's beliefs are independent of his type:

$$\mu_H(z) = \mu_L(z) \equiv \mu(z). \tag{18}$$

Therefore, a buyer's expected winning probability and his expected payments only depend on his report, but not on his type. In this case, standard arguments imply that at the optimal mechanism the incentive constraint for the high valuation buyer and the individual rationality constraint of the low valuation buyer are binding. From these binding constraints, transfers can be calculated and then inserted into the seller's objective. This delivers an expression which depends on the assignment rule alone. Finding the optimal mechanism thus amounts to finding the optimal assignment rule, which is stated in the next lemma. (The proof is in the appendix.)

Lemma 1 *Let z be given. Then an assignment rule is optimal if and only if it satisfies*

$$x^1_{HH} + x^2_{HH} = 1, \tag{19}$$

$$x^1_{HL} = x^2_{LH} = 1, \quad x^1_{LH} = x^2_{HL} = 0 \tag{20}$$

$$x^1_{LL} + x^2_{LL} = \begin{cases} 1 & if \quad \theta_L(1-p(z)) \geq \theta_H p(z) \\ 0 & if \qquad\qquad else \end{cases}. \tag{21}$$

In words, if both buyers report high valuations, then the good is assigned randomly to one of the buyers. If each buyer reports a different value, the good is assigned to the high valuation buyer, and if both buyers report low valuations, the good is assigned randomly if the ratio of the "number" of low valuation over high valuation buyers exceeds the ratio of the magnitude of the high over the low valuation, and is not assigned to any buyer otherwise. The latter is a familiar incidence of "downward distortion at the bottom".

Lemma 1 immediately delivers the seller's profit from selecting z, and thus the seller's profit from the optimal choice of z:

Lemma 2 *(a) The seller's profit from selecting z is*

$$\pi(z) \quad = \quad (1-(1-p(z))^2)\theta_H + (1-p(z))^2 \max\{0, \theta_L - \theta_H \frac{p(z)}{1-p(z)}\} - c(z). \tag{22}$$

(b) The seller's profit when he is restricted to use a pure investment strategy is

$$\pi^* \quad = \quad \max\{\pi(\bar{z}), \pi(\underline{z})\}. \tag{23}$$

The seller's profit π^* is smaller than the benchmark profit π^{FB} because if buyers' beliefs are independent, the seller has to concede an information rent to the buyers.

6 Mechanisms with mixed investment strategies

I now allow the seller to randomize over investments. Recall that ζ is the probability with which $\bar{z}$ is chosen. As mentioned earlier, buyers' valuations are now positively correlated (unless $\zeta \in \{0, 1\}$). Observing a high value indicates that a high investment level has been realized and thus increases the likelihood that the rival has observed a high value, too.

In addition to being feasible, a mechanism has now also to leave the seller indifferent between his two investment opportunities. I proceed in two steps. I first construct mechanisms which extract the ex post efficient surplus and then study conditions such that among those mechanisms one can be found that leaves the seller indifferent.

Ex post efficient surplus extracting mechanisms

The construction of surplus extracting mechanisms follows the existing literature (e.g Bose and Zhao, 2007). I consider payment rules where buyer i's payment consists of a base payment b^i_k that depends on his own report θ_k only and a contingent payment $\tau^i_{k\ell}$ which depends on the rival's announcement θ_ℓ as well. Moreover, I focus on symmetric mechanisms which treat buyers symmetrically: $b^1 = b^2$ and $\tau^1_{k\ell} = \tau^2_{\ell k}$. This allows me to consider only buyer 1 and omit the superindex i. All arguments carry over to buyer 2. The vector of contingent payments is

$$\tau_k = \begin{pmatrix} \tau_{kH} \\ \tau_{kL} \end{pmatrix}. \tag{24}$$

Extracting the ex post efficient surplus requires to implement the first best assignment rule where the buyer with the highest valuation gets the object. I assume that ties are broken by tossing a fair coin so that the ex post efficient assignment rule is given by

$$x_H = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \qquad x_L = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}. \tag{25}$$

If there is no reason for confusion, I shall drop the dependency of μ on ζ. The feasibility constraints can then be stated as follows

$$\begin{aligned} \theta_k x_k \cdot \mu_k - b_k - \tau_k \cdot \mu_k &\geq \theta_k x_\ell \cdot \mu_k - b_\ell - \tau_\ell \cdot \mu_k \quad \forall k, \ell, &(26) \\ \theta_k x_k \cdot \mu_k - b_k - \tau_k \cdot \mu_k &\geq 0 \qquad\qquad\qquad\qquad\qquad \forall k. &(27) \end{aligned}$$

To concede no information rent to buyers, these constraints have to be satisfied with (27) being binding. This can be done by choosing base payments b_k equal to a buyer's expected gross utility and designing contingent payments τ_k which (a) are orthogonal to buyer type θ_k's beliefs μ_k and (b) whose projection on the other buyer type's beliefs is sufficiently large. The latter can always be achieved if buyers' beliefs are correlated. Formally, for $k \neq \ell$:

$$b_k = \theta_k x_k \cdot \mu_k, \quad \tau_k \cdot \mu_k = 0, \quad \tau_k \cdot \mu_\ell \geq \theta_\ell x_k \cdot \mu_\ell - b_k. \tag{28}$$

The next lemma restates these conditions in terms of payment components. Base payments and the two contingent payments τ_{HH} and τ_{LL} are pinned down while the two remaining payments each have to exceed a certain threshold:

Lemma 3 *Let $\zeta \in (0, 1)$. Then there are transfers such that (28) holds. In particular,*

$$b_H = \mu_{HH}\theta_H/2 + \mu_{HL}\theta_H, \qquad b_L = \mu_{LL}\theta_L/2, \tag{29}$$

$$\tau_{HH} = -\frac{\mu_{HL}}{\mu_{HH}}\tau_{HL}, \quad \tau_{LL} = -\frac{\mu_{LH}}{\mu_{LL}}\tau_{LH}, \tag{30}$$

$$\tau_{HL} \geq \frac{\mu_{HH}}{\mu_{HH} - \mu_{LH}}[\mu_{LH}\theta_L/2 - \mu_{HH}\theta_H/2 + \mu_{LL}\theta_H - \mu_{HL}\theta_L], \tag{31}$$

$$\tau_{LH} \geq \frac{\mu_{LL}}{2(\mu_{HH} - \mu_{LH})}[\mu_{HL}\theta_H - \mu_{LL}\theta_L]. \tag{32}$$

Figure 1 illustrates the construction geometrically. The horizontal axis displays the first and the vertical axis the second component of a vector. The dashed lines indicated with the symbols $\mu_k^\perp$ are orthogonal to the buyer's belief vector μ_k. The contingent payment vectors τ_k are located on these lines and directed such that their projection on the other type's belief μ_ℓ is sufficiently large. The figure is drawn such that this occurs when τ_L points south east and stretches beyond the vertical dotted line whose location is defined by the threshold $\bar{\tau}_{LH}$; and τ_H points north west and stretches beyond the horizontal dotted line whose location is defined by the threshold $\bar{\tau}_{HL}$. In other words, the projection of contingent payments on beliefs is sufficiently large when the vector (τ_{LH}, τ_{HL}) is located in the dashed rectangle whose left lower corner is given by $(\bar{\tau}_{LH}, \bar{\tau}_{HL})$.[8]

[8]Within the running text, I write column vectors as row vectors.

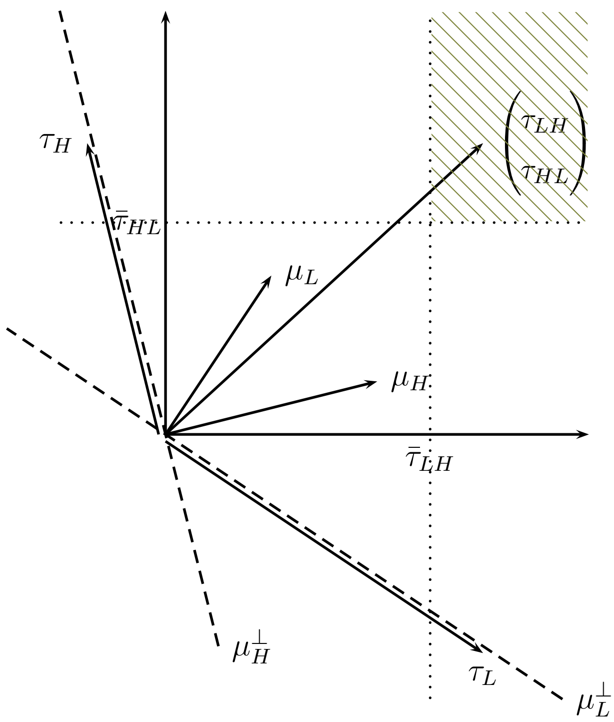

Figure 1: Construction of ex post efficient surplus extracting mechanisms.

Seller indifference

Next, I ask under what conditions transfers leave the seller indifferent between his two investment opportunities. Given the mechanism (x, t) and given that buyers' expect the seller to randomize with ζ, then the seller's profit from picking z is

$$\pi(z) = \sum_{k,\ell} [\tau_{k\ell} + b_k + \tau_{\ell k} + b_\ell] p_{k\ell}(z) - c(z) \tag{33}$$

$$= 2\{p(z)^2[\tau_{HH} + b_H] + 2p(z)(1 - p(z))[\tau_{HL} + \tau_{LH} + b_H + b_L] \tag{34}$$

$$+(1 - p(z))^2[\tau_{LL} + b_L]\} - c(z). \tag{35}$$

Hence, the seller is indifferent if $\pi(\bar{z}) = \pi(\underline{z})$, or:

$$(\bar{p}^2 - \underline{p}^2)[\tau_{HH} + b_H] + 2(\bar{p}(1 - \bar{p}) - \underline{p}(1 - \underline{p}))[\tau_{HL} + \tau_{LH} + b_H + b_L]$$
$$+[(1 - \bar{p})^2 - (1 - \underline{p})^2][\tau_{LL} + b_L] - c/2 = 0. \tag{36}$$

Ex post efficient surplus extracting mechanisms and seller indifference

The above implies that there is an equilibrium in which the seller randomizes with ζ and extracts the ex post efficient surplus if and only if there is a $\zeta \in (0,1)$ and transfers such that (30), (31), (32), (36) hold. The set of constraints can be somewhat simplified by inserting τ_{HH} and τ_{LL} from (30) in the seller's indifference constraint (36). This yields

$$\begin{aligned}\kappa_H \tau_{HL} + \kappa_L \tau_{LH} &= c/2 - (\bar{p}^2 - \underline{p}^2) b_H - 2(\bar{p}(1-\bar{p}) - \underline{p}(1-\underline{p}))(b_H + b_L) \\ &\quad -((1-\bar{p})^2 - (1-\underline{p})^2) b_L,\end{aligned} \tag{37}$$

where

$$\kappa_H = -(\bar{p}^2 - \underline{p}^2)\frac{\mu_{HL}}{\mu_{HH}} + 2(\bar{p}(1-\bar{p}) - \underline{p}(1-\underline{p})), \tag{38}$$

$$\kappa_L = 2(\bar{p}(1-\bar{p}) - \underline{p}(1-\underline{p})) - ((1-\bar{p})^2 - (1-\underline{p})^2)\frac{\mu_{LH}}{\mu_{LL}}. \tag{39}$$

In words, (37) says that the inner product of the transfer vector (τ_{HL}, τ_{LH}) and the vector $\kappa = (\kappa_H, \kappa_L)$ is equal to a certain number (which is fixed for given ζ). I summarize this observation in a lemma.

Lemma 4 *There is an equilibrium in which the seller randomizes with ζ and extracts the ex post efficient surplus if and only if there is a $\zeta \in (0,1)$ and transfers such that (31), (32), (37) hold.*

In principle, it is an algebraic matter to characterize the set of all ζ's and transfers that satisfy (31), (32), (37). Rather than determining all equilibria, I shall focus on equilibria in which the seller can almost nearly attain the ex ante efficient surplus. That is, I shall derive sufficient conditions on the set of parameters such that for arbitrarily small ϵ transfers can be found so that for $\zeta = 1 - \epsilon$ the constraints (31), (32), (37) hold.[9]

Before I turn to the condition for almost near ex ante efficiency, it is useful to think for a moment about the general case. Observe that the coefficients κ depend on ζ through the beliefs μ. Now fix ζ. Then a sufficient condition for transfers with the desired properties to exist is that

$$\kappa_H(\zeta)\kappa_L(\zeta) < 0. \tag{40}$$

To see this, recall that the constraints (31), (32) say that the transfers τ_{HL} and τ_{LH} each have to be larger than some threshold. Now suppose that at the thresholds, equation (37) is not

[9] Recall that, by assumption 1, $\bar{z}$ is the first best investment.

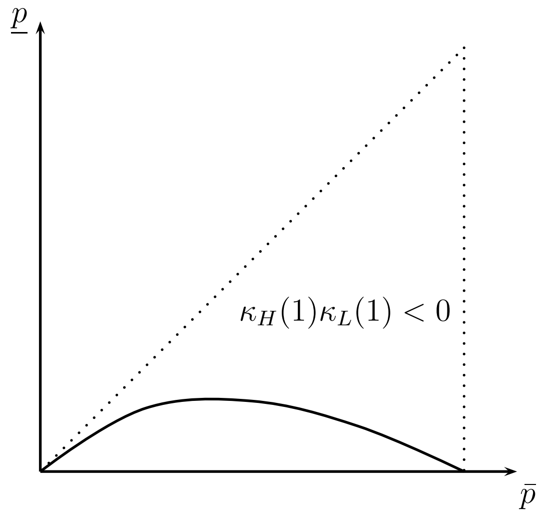

Figure 2: The set of $(\bar{p}, \underline{p})$ for which $\kappa_H(1)\kappa_L(1) < 0$.

met, say, the left hand side is larger than the right hand side. Then the left hand side can be lowered by increasing the transfer which multiplies the negative coefficient until equality holds.

Geometrically, condition (37) means that the projection of the vector (τ_{LH}, τ_{HL}) on the vector (κ_H, κ_L) has a distinct length. (Recall that the inner product is proportional to the projection.) Thus, to satisfy (31), (32) and (37) jointly, a vector in the dashed rectangle has to be found whose projection on (κ_H, κ_L) equals this length. It is evident that this can be achieved if the vector (κ_H, κ_L) is located in the north-east or the south-west quadrant. But this is equivalent to the condition $\kappa_H \kappa_L < 0$.

I now examine when $\kappa_H(\zeta)\kappa_L(\zeta) < 0$ for ζ close to one. Because of continuity, if the inequality is true for $\zeta = 1$, then it will still be true for ζ close to one. The next lemma characterizes when $\kappa_H(1)\kappa_L(1) < 0$ in terms of $\bar{p}$ and $\underline{p}$.

Lemma 5 *It holds:*

$$\kappa_H(1)\kappa_L(1) < 0 \qquad \Leftrightarrow \qquad \underline{p} > \frac{\bar{p}(1-\bar{p})}{1+\bar{p}}. \tag{41}$$

Figure 2 illustrates Lemma 5 graphically. The area between the 45 degree and the curved line is the set of parameters $\bar{p}$ and $\underline{p}$ for which $\kappa_H(1)\kappa_L(1) < 0$.

I can finally turn to the seller's optimal choice of mechanism at stage 1. Under the conditions of Lemma 5 there is an (arbitrarily small) ϵ and a mechanism such that it is an equilibrium that, at stage 2, the seller chooses $\bar{z}$ with probability $1-\epsilon$ and $\underline{z}$ with probability ϵ and, at stage 3, buyers accept the mechanism and announce their types truthfully. Since the seller fully extracts the ex post efficient surplus, his ex ante profit from this mechanism is

$$\pi^{FB}(\bar{z})(1-\epsilon) + \pi^{FB}(\underline{z})\epsilon. \tag{42}$$

Thus, since $\bar{z}$ is the first best investment level by assumption 1, I conclude:

Proposition 1 *Suppose the conditions of Lemma 5 hold. Then the seller can attain nearly the first best profit.*

7 Extensions

7.1 More than two investments

I now allow for M possible investments z_m which the seller can choose at a cost c_m.[10] Conditional on investment z_m, the probability with which a buyer observes a high valuation is p_m, where, as before, valuations are independent conditional on investments. The joint probability that $(\theta_1, \theta_2) = (\theta_k, \theta_\ell)$ occurs conditional on z_m is denoted by $p^m_{k\ell}$. Assume that p_m is increasing in m.

Suppose there is a unique first best investment level $z_{\bar{m}}$ with the index $\bar{m}$ given by

$$\bar{m} = \arg\max_m \sum_{k,\ell} \max\{\theta_k, \theta_\ell\} p^m_{k\ell} - c_m. \tag{43}$$

I shall also write $\bar{z}$ for $z_{\bar{m}}$ and $\bar{p}$ for $p_{\bar{m}}$. To reduce the number of case distinction, I assume that $\bar{p}$ is "interior", that is, it is neither the smallest nor the largest p_m:

Assumption 2: $p_{\bar{m}-1} < \bar{p} < p_{\bar{m}+1}$.

The construction follows closely the construction in the previous section. I seek for sufficient conditions under which the seller can extract nearly the full efficient surplus by choosing the efficient investment with almost full probability mass and choosing one single different investment level with the remaining mass. With more than two investment levels, however, equilibrium requires that the seller cannot gain by choosing one of the remaining pure investment levels. Thus, the question is if transfers can be found that satisfy this additional equilibrium requirement.

[10] Below, I comment on extensions with respect to types θ.

Suppose the seller chooses the first best investment $\bar{z}$ with probability ζ and some other pure investment $z_{\hat{m}}$ with probability $1-\zeta$. Which of the investments $z_{\hat{m}}$ does the job to achieve almost full surplus extraction will be determined below. Denote this strategy by $I(\zeta, \hat{m})$.

With the appropriate notational modifications, (30), (31), (32) still describe the conditions for an ex post efficient surplus extracting mechanism if buyers believe that the seller adopts $I(\zeta, \hat{m})$. For the seller to actually adopt $I(\zeta, \hat{m})$, he has to be indifferent between $\bar{z}$ and $z_{\hat{m}}$ and prefer $\bar{z}$ over each z_m. Again, with the appropriate notational modifications, the indifference condition is given by (36) and the preference conditions are given by (36) with the equality replaced by a weak inequality. As above we can insert the constraint (30) in the indifference and preference constraints. This yields

$$\begin{aligned}\kappa_H^{\hat{m}}\tau_{HL} + \kappa_L^{\hat{m}}\tau_{LH} &= (c_{\bar{m}} - c_{\hat{m}})/2 - (\bar{p}^2 - p_{\hat{m}}^2)b_H - 2(\bar{p}(1-\bar{p}) - p_{\hat{m}}(1-p_{\hat{m}}))(b_H + b_L) \quad (44)\\ &\quad -((1-\bar{p})^2 - (1-p_{\hat{m}})^2)b_L,\\ \kappa_H^{m}\tau_{HL} + \kappa_L^{m}\tau_{LH} &\geq (c_{\bar{m}} - c_{m})/2 - (\bar{p}^2 - p_{m}^2)b_H - 2(\bar{p}(1-\bar{p}) - p_{m}(1-p_{m}))(b_H + b_L) \quad (45)\\ &\quad -((1-\bar{p})^2 - (1-p_{m})^2)b_L.\end{aligned}$$

where for all $m \neq \bar{m}$:

$$\kappa_H^m = -(\bar{p}^2 - p_m^2)\frac{\mu_{HL}}{\mu_{HH}} + 2(\bar{p}(1-\bar{p}) - p_m(1-p_m)), \tag{46}$$

$$\kappa_L^m = 2(\bar{p}(1-\bar{p}) - p_m(1-p_m)) - ((1-\bar{p})^2 - (1-p_m)^2)\frac{\mu_{LH}}{\mu_{LL}}. \tag{47}$$

In words, (44) says that the inner product of the vector (τ_{LH}, τ_{HL}) and the vector $\kappa^{\hat{m}} = (\kappa_H^{\hat{m}}, \kappa_L^{\hat{m}})$ is equal to a certain number. (45) says that the inner product of the vector (τ_{LH}, τ_{HL}) and the vector $\kappa^m = (\kappa_H^m, \kappa_L^m)$ exceeds a certain threshold. The next lemma summarizes.

Lemma 6 *There is an equilibrium in which the seller adopts $I(\zeta, \hat{m})$ and extracts the ex post efficient surplus if and only if there is a $\zeta \in (0,1)$ and transfers such that (31), (32), (44), (45) hold.*

Next, I aim at a sufficient condition for the lemma to be satisfied. To fix language, consider an arbitrary vector v which is neither null nor perfectly horizontal. The line trough v partitions the plane into two half spaces. I refer to the "right half space of v" as the half space that is located to the right of that line. Formally the right half space of v is the set $\{v' \in \mathbb{R}^2 \mid v' = \lambda v + \lambda'(1,0)$ for some $\lambda \in \mathbb{R}$ and $\lambda' > 0\}$.

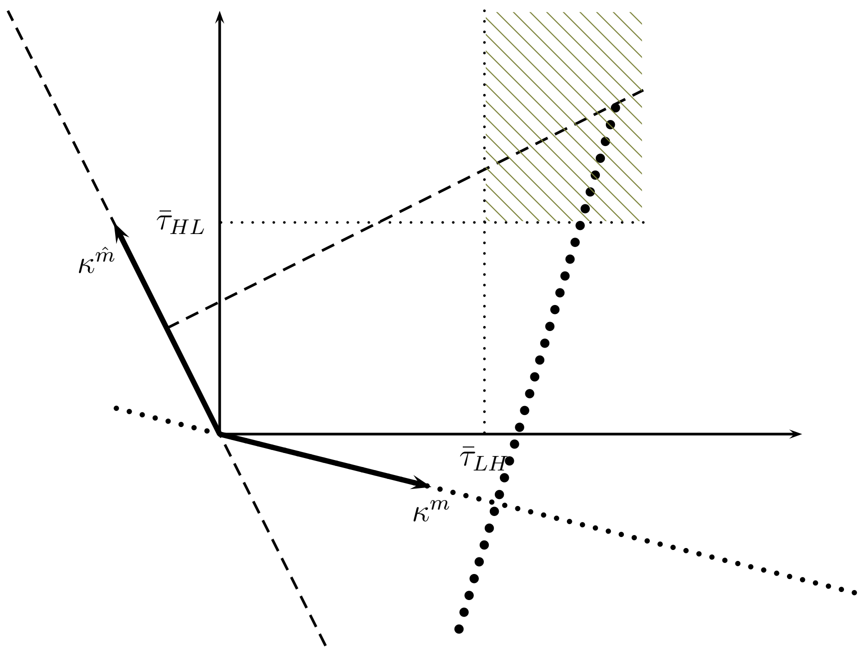

Figure 3: Equilibrium construction for more than two investments.

The sufficient condition is as follows: There is an $\hat{m}$ and a ζ such that

$$\kappa_H^{\hat{m}}(\zeta)\kappa_L^{\hat{m}}(\zeta) < 0, \quad and \tag{48}$$

$$\kappa^m(\zeta) \text{ is in the right half space of } \kappa^{\hat{m}}(\zeta) \quad \forall m \neq \hat{m}, \bar{m}. \tag{49}$$

To see why this is sufficient, suppose first that $\kappa_H^{\hat{m}}(\zeta)\kappa_L^{\hat{m}}(\zeta) < 0$. As in the previous section, this guarantees that transfers can be found such that (31), (32), (44) hold. In Figure 3, the vector $\kappa^{\hat{m}}$ points to the north west. To satisfy (31), (32), the transfer vector (τ_{HL}, τ_{LH}) has to be in the shaded rectangle, and to satisfy (44) the projection of transfers on $\kappa^{\hat{m}}$ has to be of a distinct length. In the figure, this means that transfers are located on the dashed line, which is orthogonal to $\kappa^{\hat{m}}$. Thus, all transfers on the dashed line segment in the shaded rectangle satisfy (31), (32), (44).

Next, suppose that the vector κ^m is located in the right half space of $\kappa^{\hat{m}}$. Recall that (45) says that the projection of the transfer vector (τ_{HL}, τ_{LH}) on the vector $\kappa^m = (\kappa_H^m, \kappa_L^m)$ is sufficiently large. In Figure 3, one such projection is indicated by the thickly dotted line and is given by the distance between the origin and the intersection of the two dotted lines. Observe that by raising transfers along the dashed line segment in the shaded rectangle this projection can be made arbitrarily large until (45) is met.

I now proceed analogously to the previous section. I shall look for conditions on the primitives p_m such that (48) and (49) hold for $\zeta = 1$. Since (48) and (49) are inequality conditions, continuity implies that they still hold for $\zeta = 1 - \epsilon$ for arbitrarily small ϵ. Thus, there will be an equilibrium in which the seller adopts the strategy $I(1-\epsilon, \hat{m})$ and attains nearly the first best surplus:

Proposition 2 *Suppose that*

$$p_{\bar{m}-1} > \frac{\bar{p}(1-\bar{p})}{1+\bar{p}} \quad \textit{or} \quad p_{\bar{m}+1} < \frac{2(1-\bar{p})+\bar{p}^2}{2-\bar{p}}. \tag{50}$$

Then the conditions (48) and (49) with $\zeta = 1$ hold for $\hat{m} = \bar{m}-1$ or $\hat{m} = \bar{m}+1$. Thus, the seller can nearly attain the first best profit.

The proof of the lemma shows that the set of vectors κ^m, $m \neq \bar{m}$ are located within a fan enclosed by the two vectors $\kappa^{\bar{m}-1}$ and $\kappa^{\bar{m}+1}$. Moreover, the latter two vectors enclose an angle of less than 180°. This implies that (49) is satisfied for both $\hat{m} = \bar{m}-1$ and $\hat{m} = \bar{m}+1$. Thus, the remaining question is when (48) is met for one of these $\hat{m}$'s. The answer is given by the algebraic condition stated in the lemma.

Figure 4 illustrates the parameter condition in Proposition 2 graphically. The condition requires $p_{\bar{m}-1}$ be between the dashed and the 45 degree line, or $p_{\bar{m}+1}$ be located between the 45 degree and the dotted line. (Recall that by assumption 2: $p_{\bar{m}-1} < \bar{p} < p_{\bar{m}+1}$.)

7.2 More than two buyer types

I conclude this section with a brief discussion on the extension to more than two buyer valuation types. Consider a symmetric environment with K valuations $\theta_{i1}, \ldots, \theta_{iK}$ and return to the case with two investment levels (the extension to more investments works then as above). Any mixing probability ζ induces, in equilibrium, for each buyer type a belief about rival buyers' types. Cremer and McLean (1988) have shown that ex post surplus extracting mechanisms exist if no type's belief is in the convex hull of the other types' beliefs. This condition will be typically satisfied for $\zeta \in (0,1)$. The (symmetric) mechanism involves K^2 transfers, one for

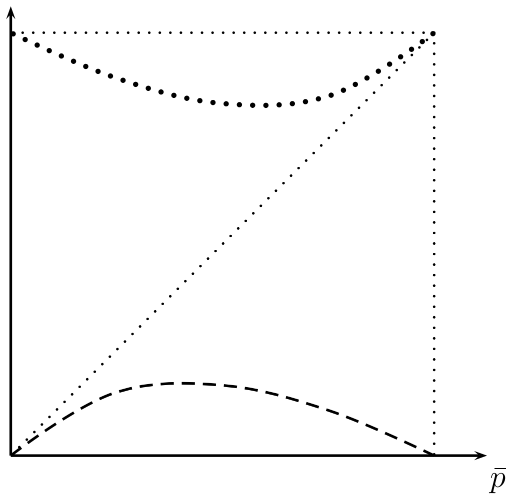

Figure 4: The curves $\frac{\bar{p}(1-\bar{p})}{1+\bar{p}}$ (dashed) and $\frac{2(1-\bar{p})+\bar{p}^2}{2-\bar{p}}$ (dotted).

each pair of buyer types. The participation constraints determine K transfers, and the incentive compatibility constraints impose $K(K-1)$ linear inequality constraints on the remaining transfers.

Those inequality constraints have to be satisfied jointly with the indifference constraint for the seller. The indifference constraint is given by a single equation which is linear in the $K(K-1)$ transfers. Thus, by naively counting equations and variables, it should become easier to satisfy the indifference constraint the higher the number of available transfers, i.e. the higher the number of types.

8 Conclusion

The idea that the seller randomizes to create correlation may potentially also be applied in a bilateral trade context. Schmitz (2002) studies a bilateral trade model which differs from mine in two dimensions: first, there is only one buyer, and second, the buyer has to agree to the contract before the seller invests and cannot reject it anymore after having learnt his valuation. Schmitz establishes an inefficiency result which rests on the insight that incentive compatibility requires the buyer to obtain an information rent ex post. In fact, the second best welfare maximizing mechanism displays an inefficient trade rule but an efficient payment rule (i.e. is budget balanced ex post). In Schmitz' framework, the probability of trade and payments depend only on a report by the buyer about his valuation, yet not by the seller about his investment. Now imagine the seller randomizes over investments, then the two parties hold correlated private information. Hence, a mechanism of the Cremer and McLean (1988) type, which conditions on reports by the buyer and the seller, could, in principle, elicit this information without conceding information rents to the buyer, i.e. would display an efficient trade rule. The problem in a bilateral trade context is, however, that such a mechanism would involve fining both parties in some states.[11] In the absence of a budget breaker, parties would therefore need to inefficiently burn money, precluding the first best. Still, one may ask if such a mechanism could improve upon a mechanism that conditions on the buyer's information only.

Appendix

Proof of Lemma 1 Since the incentive constraint binds for the high valuation buyer and the individual rationality constraint binds for the low valuation buyer, expected payments for the high and the low valuation buyer are:

$$\sum_{m\in\{L,H\}} t^i_{Hm}\mu_m(\zeta) = \theta_H \sum_{m\in\{L,H\}} x^i_{Hm}\mu_m(\zeta) - \theta_H \sum_{m\in\{L,H\}} x^i_{Lm}\mu_m(\zeta) + \sum_{m\in\{L,H\}} t^i_{Lm}\mu_m(\zeta) \tag{51}$$

$$\sum_{m\in\{L,H\}} t^i_{Lm}\mu_m(\zeta) = \theta_L \sum_{m\in\{L,H\}} x^i_{Lm}\mu_m(\zeta) \tag{52}$$

With this, the expected sum of transfers becomes:

$$R(z) = \sum_{k,\ell}[t^1_{k\ell} + t^2_{k\ell}]p_{k\ell}(z) \tag{53}$$

$$\begin{aligned} &= \theta_H x^1_{HH}p_{HH}(z) + \theta_H x^1_{HL}p_{HL}(z) - \theta_H x^1_{LH}p_{HH}(z) - \theta_H x^1_{LL}p_{HL}(z) \\ &\quad +\theta_L x^1_{LH}p_{LH}(z) + \theta_L x^1_{LL}p_{LL}(z) + \theta_H x^2_{HH}p_{HH}(z) + \theta_H x^2_{LH}p_{LH}(z) \\ &\quad -\theta_H x^2_{HL}p_{HH}(z) - \theta_H x^2_{LL}p_{LH}(z) + \theta_L x^2_{HL}p_{HL}(z) + \theta_L x^2_{LL}p_{LL}(z) \end{aligned} \tag{54}$$

Due to conditional independence, it holds that $p_{HL}(z) = p_{LH}(z)$, and thus:

$$R(z) = \theta_H[x^1_{HH} + x^2_{HH}]p_{HH}(z) + \tag{55}$$

$$\begin{aligned} &x^1_{LH}(-\theta_H p_{HH} + \theta_L p_{LH}) + x^2_{LH}\theta_H p_{LH} + \\ &x^1_{HL}\theta_H p_{HL} + x^2_{HL}(-\theta_H p_{HH} + \theta_L p_{HL}) + \\ &x^1_{LL}(-\theta_H p_{HL} + \theta_L p_{LL}) + x^2_{LL}(-\theta_H p_{HL} + \theta_L p_{LL}). \end{aligned} \tag{56}$$

[11] In fact, with two parties and correlated values, no full surplus extracting mechanism exists which is ex post budget balanced (see d'Aspremont et al., 2004).

It is now easy to verify that the optimal allocation is as stated in the lemma. □

Proof of Lemma 2 Use the optimal assignment rule (19)-(21) in (55). □

Proof of Lemma 3 The equalities in (29) resp. (30) are immediate transformations of the first two equalities in (28). To see (31), write out the right inequality in (28) for $k = H, \ell = L$:

$$\tau_{HH}\mu_{LH} + \tau_{HL}\mu_{LL} \geq m_H - \theta_L(\mu_{LH}/2 + \mu_{LL}). \tag{57}$$

Plug in $\tau_{HH} = -\frac{\mu_{HL}}{\mu_{HH}}\tau_{HL}$ and solve for τ_{HL}:

$$\begin{aligned} \tau_{HL}(-\frac{\mu_{HL}}{\mu_{HH}}\mu_{LH} + \mu_{LL}) &\geq \theta_L(\mu_{LH}/2 + \mu_{LL}) - m_H && (58)\\ &= \mu_{LH}\theta_L/2 - \mu_{HH}\theta_H/2 + \mu_{LL}\theta_L L - \mu_{HL}\theta_H. && (59) \end{aligned}$$

Expression (31) now follows by noting that

$$-\frac{\mu_{HL}}{\mu_{HH}}\mu_{LH} + \mu_{LL} = \frac{\mu_{HH} - \mu_{LH}}{\mu_{HH}} > 0. \tag{60}$$

The inequality (32) can be derived analogously. □

Proof of Lemma 5 The claim follows from the following two properties:

$$\begin{aligned} \kappa_H(1) < 0 \qquad &\Leftrightarrow \quad \underline{p} > \frac{\bar{p}(1-\bar{p})}{1+\bar{p}}, && (61)\\ \kappa_L(1) > 0 \qquad &\text{for all } \bar{p}, \underline{p}. && (62) \end{aligned}$$

To see (61), note that for $\zeta = 1$ it holds that $\mu_{HL} = 1 - \bar{p}$ so that

$$\begin{aligned} \kappa_H(1) &= -(\bar{p}^2 - \underline{p}^2)\frac{1-\bar{p}}{\bar{p}} + 2(\bar{p}(1-\bar{p}) - \underline{p}(1-\underline{p})) && (63)\\ &= -(\bar{p} - \underline{p})[\frac{1+\bar{p}}{\bar{p}}\underline{p} - (1-\bar{p})]. && (64) \end{aligned}$$

Since $\bar{p} > \underline{p}$, this implies (61).

To see (62), note that for $\zeta = 1$ it holds that $\mu_{LH} = \bar{p}$. Now use the notation $\bar{r} = 1 - \bar{p}$ and $\underline{r} = 1 - \underline{p}$. Then:

$$\begin{aligned} \kappa_L(1) &= -(\bar{r}^2 - \underline{r}^2)\frac{1-\bar{r}}{\bar{r}} + 2(\bar{r}(1-\bar{r}) - \underline{r}(1-\underline{r})) && (65)\\ &= -(\bar{r} - \underline{r})[\frac{1+\bar{r}}{\bar{r}}\underline{r} - (1-\bar{r})]. && (66) \end{aligned}$$

Since $\bar{r} < \underline{r}$, it follows that $\kappa_L(1) > 0$ if and only if $\underline{r} > \frac{\bar{r}(1-\bar{r})}{1+\bar{r}}$. But this is always true since $\bar{r} > \underline{r} > \frac{\bar{r}(1-\bar{r})}{1+\bar{r}}$. □

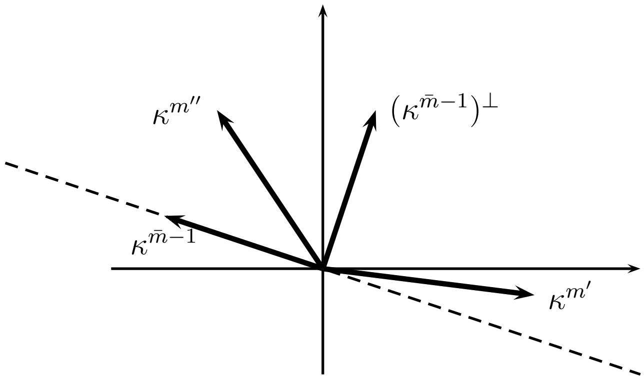

Figure 5: Location of $\kappa^{\bar{m}-1}$ and its orthogonal vector $(\kappa^{\bar{m}-1})^{\perp}$.

Proof of Proposition 2 Consider first the case $p_{\bar{m}-1} > \frac{\bar{p}(1-\bar{p})}{1+\bar{p}}$. I demonstrate that (48) and (49) hold for $\hat{m} = \bar{m}-1$ and $\zeta = 1$. (For the rest of the proof, I fix $\zeta = 1$ omit it as an argument in κ.)

As for (48). The argument is the same as in the proof of Lemma 5 where $p_{\bar{m}-1}$ now plays the role of $\underline{p}$ then.

As for (49). We first remark that as in the proof of Lemma 5, $p_{\bar{m}-1} > \frac{\bar{p}(1-\bar{p})}{1+\bar{p}}$ implies that $\kappa_H^{\bar{m}-1} < 0$ and $\kappa_L^{\bar{m}-1} > 0$. Now, observe that the vector κ^m is located in the right half space of $\kappa^{\bar{m}-1}$ if and only if κ^m has a positive projection on that orthogonal vector of $\kappa^{\bar{m}-1}$ which is in the right half space of $\kappa^{\bar{m}-1}$. With the first remark, this orthogonal vector is given by $(\kappa_L^{\bar{m}-1}, -\kappa_H^{\bar{m}-1})$. Therefore, κ^m is located in the right half space of $\kappa^{\bar{m}-1}$ if and only if

$$\kappa_L^{\bar{m}-1}\kappa_H^m - \kappa_H^{\bar{m}-1}\kappa_L^m > 0. \tag{67}$$

Figure 5 illustrates the argument graphically.

I now demonstrate that (67) is true. Indeed, as in (64) and (66), I can write for all m:

$$\kappa_H^m = -(\bar{p} - p_m)[\frac{1+\bar{p}}{\bar{p}}p_m - (1-\bar{p})], \tag{68}$$

$$\kappa_L^m = (\bar{p} - p_m)[\frac{2-\bar{p}}{1-\bar{p}}(1-p_m) - \bar{p}]. \tag{69}$$

Therefore, the left hand side in (67) becomes:

$$-(\bar{p} - p_{\bar{m}-1})(\bar{p} - p_m) \quad \{ \quad \frac{2-\bar{p}}{1-\bar{p}}\frac{1+\bar{p}}{\bar{p}}(1-p_{\bar{m}-1})p_m - (2-\bar{p})(1-p_{\bar{m}-1}) - (1+\bar{p})p_m \tag{70}$$

$$+\bar{p}(1-\bar{p}) \tag{71}$$

$$-\frac{2-\bar{p}}{1-\bar{p}}\frac{1+\bar{p}}{\bar{p}}p_{\bar{m}-1}(1-p_m)+(1+\bar{p})p_{\bar{m}-1}+(2-\bar{p})(1-p_m) \quad (72)$$
$$-\bar{p}(1-\bar{p}) \;\; \}. \quad (73)$$

Re–arranging delivers

$$-(\bar{p}-p_{\bar{m}-1})(\bar{p}-p_m)\{\frac{2-\bar{p}}{1-\bar{p}}\frac{1+\bar{p}}{\bar{p}}(p_m-p_{\bar{m}-1})-(2-\bar{p})(p_m-p_{\bar{m}-1}) \quad (74)$$
$$-(1+\bar{p})(p_m-p_{\bar{m}-1})\} \quad (75)$$
$$= \quad -(\bar{p}-p_{\bar{m}-1})(\bar{p}-p_m)(p_m-p_{\bar{m}-1})\{\frac{2-\bar{p}}{1-\bar{p}}\frac{1+\bar{p}}{\bar{p}}-3\}. \quad (76)$$

It is easy to verify that the term in the curly brackets is strictly positive. Now, if $p_m < p_{\bar{m}-1} < \bar{p}$, then the terms in the first two round brackets are positive and the term in the third round brackets is negative, so that the overall expression is positive. If $p_m > p_{\bar{m}-1}$ and thus, by assumption 2, also $p_m > \bar{p}$, then the terms in the first and third round brackets are positive and the term in the second round brackets is negative, so that again the overall expression is positive. This establishes (49) for the case $p_{\bar{m}-1} > \frac{\bar{p}(1-\bar{p})}{1+\bar{p}}$.

Next, consider the case $p_{\bar{m}+1} < \frac{2(1-\bar{p})+\bar{p}^2}{2-\bar{p}}$. I demonstrate that (48) and (49) hold for $\hat{m} = \bar{m}+1$ and $\zeta = 1$.

As for (48). The claim follows from the following two properties:

$$\kappa_H^{\bar{m}+1} \; > \; 0, \quad (77)$$
$$\kappa_L^{\bar{m}+1} \; > \; 0 \quad \Leftrightarrow \quad p_{\bar{m}+1} < \frac{2(1-\bar{p})+\bar{p}^2}{2-\bar{p}}. \quad (78)$$

To see (77), use (68) to write

$$\kappa_H^{\bar{m}+1} \;=\; -(\bar{p}-p_{\bar{m}+1})[\frac{1+\bar{p}}{\bar{p}}p_{\bar{m}+1}-(1-\bar{p})]. \quad (79)$$

The term in the square brackets is positive if $p_{\bar{m}+1} > \frac{(1-\bar{p})\bar{p}}{1+\bar{p}}$, but this is always true since $\frac{(1-\bar{p})\bar{p}}{1+\bar{p}} > \bar{p}$ and $p_{\bar{m}+1} > \bar{p}$ by assumption 2, and this establishes (77).

To see (78), use (69) to write

$$\kappa_L^{\bar{m}+1} \;=\; (\bar{p}-p_{\bar{m}+1})[\frac{2-\bar{p}}{1-\bar{p}}(1-p_{\bar{m}+1})-\bar{p}]. \quad (80)$$

Since $p_{\bar{m}+1} > \bar{p}$, this expression is positive if the term in the squared brackets is negative. It is easy to verify that the latter is the case if and only if the condition in (78) holds.

As for (49). The argument is analogous to the argument used to establish (49) in the first case. This completes the proof. □

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